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A COURSE OF

INTEGRATIONAL AND OPERATIONAL CALCULUS

WITH APPLICATIONS TO PROBLEMS OF PHYSICS AND ELECTROTECHNICS

BY

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PREFACE.

In physics and technics one is often faced with the problem of studying the periodical steady state of a mechanical or electrical system, corresponding to a sinusoidal disturbance attacking it. If the system is governed by linear differential equations with coefficients, not depending upon time, one can, as we know, by introducing the representative complex vectors of the time functions obtain new, simpler equations, which do not contain the variable of time. This mode of proceeding has, especially in electrotechnics, got an extraordinary practical importance, being the basis of the so-called symbolic method in the alternating-current theory.

It seems to be O. Heaviside who first systematically worked out methods of treating non-periodical and transient phenomena in a symbolic way. In later years Heaviside's so-called operational calculus has been subject to an increasing interest, and several attempts have been made to found the many operations occurring upon a more general and mathematically more satisfactory basis. Hence, the literature about operational calculus is nowadays rather copious. A selected collection of textbooks and recent treatises on the subject is given at the end of this book.

The aim of the present work is to give an account of certain methods of integration, nearly related to the Heaviside operational calculus but more consequently building upon the symbolic method of the alternating-current theory. The author has tried to give a relatively simple but mathematically well-founded generalization of the principal ideas and notions of this method. Of course, the Fourier integral theorem offers a natural way to gain this aim, but, like D. P. Dalzell, among others, the author has found the modification of this theorem due to Mellin more adequate for the purpose, and thus based the development of the methods upon that form of the theorem. This will cause ordinary analytical methods of the theory of functions of a complex variable to play a prominent part in the treatment. This will also be the case with some procedures of the functional calculus, because there exists a near relationship between operational methods and this branch of analysis. Indeed, as outlined by P. Lévy, it is possible to build up most of Heaviside's operational calculus upon the functional calculus as a basis.

In the course of the work the author has been influenced by many other treatises, to mention only the wellknown textbooks by Jeffreys, Carson, Bush and Berg. As to the practical applications these books and the present one complete one another in many respects. It ought to be noticed that in the following treatise the main interest concerns the methodical and mathematical side of the subject; therefore a more detailed discussion of the solutions and especially of their physical contents is generally let aside. It has been my intention to present many results under such analytic forms as usually occur in other textbooks, and the reader is once for all referred to standard works such as those by Riemann-Weber, Webster and Courant-Hilbert besides the books already mentioned.

When trying to develop consequently the operational calculus as an extension of the symbolic method the author got reason to introduce the notion of integrator, essentially corresponding to the representative complex vector of a sinusoidal function; this brings also with it a certain difference from other textbooks in the mode of writing operational equations. However, also the notion of operator has been frequently used, mostly signifying a linear functional operation, and therefore it seems justified to employ the name *integrational and operational calculus* on the methods to be presented.

Some proofs and supplements of a more completing or special nature have been transferred to an Appendix at the end of the text.

This course is essentially meant for electrical engineers that do not content themselves with a purely formal management of the operational calculus. Of course, it does not claim at all upon completeness, but if, when studied at the side of other similar books, it could give the reader some new and useful aspects on the subject, its aim would be gained sufficiently well.

Örebro, Aug. 1935.

K. D.

A course of
Integrational and Operational Calculus
with applications to problems of
Physics and Electrotechnics.

1. The integral theorem. General points of view.

In the following pages we shall have to deal with the important problem of calculating the effects of arbitrary external forces attacking given mechanical or electrical systems. We assume the dynamical processes to be governed by differential equations that are linear and have constant coefficients with respect to the variable of time t. The physical sense of the linearity is the validity of the *principle of superposition*, expressing that the effects of different forces will independently be added to one another. Hence our problem essentially consists in computing the effect of a single force attacking the system at any moment; the influence of certain prescribed initial conditions may also be included in this problem. Nothing prevents us from taking still one step towards a simplification, assuming the external force to keep the constant value 1; indeed, forces of any nature may easily be built up by elements of that type, represented by the discontinuous unit function or »jump function» $H(t)$.

Accordingly, we are going to work with a general class of functions possessing the characteristic property of being zero identically up to a certain moment, say $t=0$, when they change, in general discontinuously, into functions more or less regular for positive values of t.

The practical methods of integration depend upon the possibility of *algebraizing* the expressions in the time-derivatives and -integrals. In that way, instead of ordinary differential equations in t we get purely algebraical equations, and partial differential equations in t and other variables x, y, z..... are replaced by simpler equations not containing t. To permit the algebraization the time function in question must be decomposed into a number of exponential elements. Generally the external forces will be of such a nature that the classical Fourier integral theorem can not be applied, but a variant af this theorem, due to Mellin, will prove to be very suitable for the purpose.

The *Fourier-Mellin theorem* can be formulated in the following way, adjusted with respect to our special state of things (Cf. Appendix 1):

Proposed the real function $f(t)$ possesses the following properties:

1. $f(t)$ and its first derivative are piecewise continuous in each finite interval.

2. $f(t) = \frac{1}{2}\,[f(t+0)+f(t-0)]$ at a point of discontinuity.

3. $\int\limits_{-\infty}^{+\infty} |f(\tau)|\cdot e^{-\sigma\tau}\cdot d\tau$ exists, when $\alpha_1 < \sigma < \alpha_2$.

Then $f(t)$ may be represented by the complex integral

$$f(t)=\frac{1}{2\pi i}\cdot\int\limits_{\sigma-i\infty}^{\sigma+i\infty} F(p)\cdot\frac{e^{pt}}{p}\cdot dp, \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

where

$$F(p)=p\cdot\int\limits_{-\infty}^{+\infty} f(\tau)\cdot e^{-p\tau}\cdot d\tau. \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

In (1) we assume the path of integration to be a straight line parallel to the imaginary axis of the p-plane. This line, however, can be arbitrarily removed within the domain

$$\alpha_1 < Re\,\{p\} < \alpha_2$$

without changing the value of the integral; it may also be replaced by any curve in this band between the extremities $\sigma \pm i\,\infty$ (Cf. fig. 1).

Without essentially limiting the practical applicability of our methods we may generally presume more special properties of the function $f(t)$ than those required by the integral theorem. Hence,

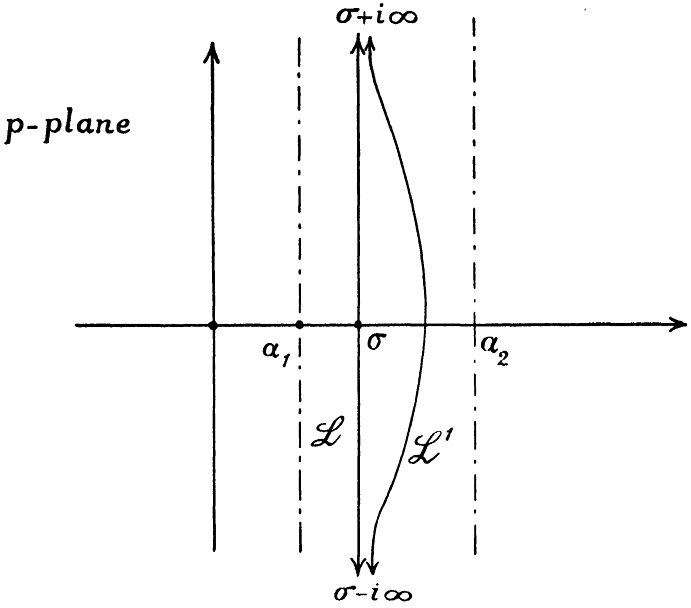

Fig. 1.

as already mentioned, we can assume that $f(t) = 0$, when $t < 0$, and accordingly put the lower limit in (2) equal to zero. Further, for positive values of t we will prescribe $f(t)$ to be continuous and to have a generally continuous and exponentially limited derivative or sometimes such derivatives of still higher orders.

A function $\varphi(t)$ is said to be *exponentially limited* when $t \to +\infty$, if the inequality $|\varphi(t)| < e^{\alpha t}$ holds for all $t > t_*$, the real exponent α being independent of t.

To simplify our mode of expression we say that a function $f(t)$ *belongs to class* n if, for positive values of t, it is continuous and possesses continuous derivatives up to the order $(n-1)$ and an at least piecewise continuous and exponentially limited n:th derivative. Then, as is readily proved, there must exist a number α and a number t_* such that

$$|f^{(k)}(t)| < e^{\alpha t} \text{ when } t > t_* \quad \ldots\ldots\ldots\ldots \quad (3)$$
$$(k = 0, 1, \ldots n)$$

With the notation $f(0)$ we always understand the limit value of $f(\varepsilon)$ when $\varepsilon \to +0$.

A function is said to belong to *the class of order infinity* if it has an unlimited sequence of continuous, uniformly bounded, and uniformly exponentially limited derivatives.

The latter type of functions is most common in the applications. When not otherwise stated we will presume the time-functions appearing in the following to belong to class nr 1 at least.

On account of this statement the interval $\alpha_1 < \sigma < \alpha_2$ in the integral theorem may be replaced by an infinite interval : $\sigma > \alpha$. Here α has the same meaning as in (3). Hence, the line of integration L (Cf. fig. 1) can be situated anywhere to the right of the line $Re\ \{p\} = \alpha$.

It is easily verified (Cf. Appendix 3) that a function $F(p)$, being defined in accordance with (2) by any function $f(t)$, has the following general properties:

1. Within the half-plane $Re\ \{p\} > \alpha$ the function $\frac{F(p)}{p}$ is holomorphical.

2. Within the half-plane $Re\ \{p\} \geq \sigma_0 > \alpha$ the function $\frac{F(p)}{p}$ tends uniformly to zero, when $|p| \to \infty$.

Some simple examples may illustrate the subdivision of a function $f(t)$ into exponential elements by the integral theorem.

A. *The unit-function $H(t)$.*

This fundamental function is the simplest in the class of functions in question. It is defined by:

$$H(t) = \begin{cases} 0 & t<0. \\ (\tfrac{1}{2} & t=0.) \\ 1 & t>0. \end{cases}$$

The value for $t=0$, expressing a regular discontinuity at the origin, is immaterial for us. According to our agreement above we always put $H(0)=1$.

For this function we get by (2):

$$F(p) = p \cdot \int_0^\infty e^{-p\tau} \cdot d\tau = \left[-e^{-p\tau}\right]_0^\infty = 1$$

if $\sigma > 0$, and accordingly, by (1):

$$H(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{pt}}{p} \cdot dp \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

for any line of integration on the right of the imaginary axis.

If $\varphi(t)$ denotes a function defined for any real value of t, e. g. by an analytic expression, we get, when multiplying by the unit-function, a new function, $\varphi(t) \cdot H(t)$, which is zero for all negative values of t and coincides with $\varphi(t)$ for positive values of t.

B. *The exponential function* $e^{\alpha t} \cdot H(t)$ (α *real*).

One gets the formulas:

$$\left.\begin{aligned} F(p) &= p \cdot \int_0^\infty e^{(\alpha - p) \cdot \tau} \cdot d\tau = \frac{p}{p - \alpha} \cdot \\ e^{\alpha t} \cdot H(t) &= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{p}{p - \alpha} \cdot \frac{e^{pt}}{p} \cdot dp. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (5)$$

provided that $\sigma > \alpha$.

Note. When α is negative, we also have $\sigma = 0 > \alpha$; hence the imaginary axis may be used as a path of integration, which means that the classical Fourier theorem is applicable.

C. *The functions* $\sin \omega t \cdot H(t)$ *and* $\cos \omega t \cdot H(t)$.

If $\sigma > 0$ we get from the integral theorem

$$\sin \omega t \cdot H(t) = \frac{1}{2\pi i} \cdot \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\omega p}{p^2 + \omega^2} \cdot \frac{e^{pt}}{p} \cdot dp, \quad \ldots\ldots\ldots\ldots \quad (6)$$

$$\cos \omega t \cdot H(t) = \frac{1}{2\pi i} \cdot \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{p^2}{p^2 + \omega^2} \cdot \frac{e^{pt}}{p} \cdot dp. \quad \ldots\ldots\ldots\ldots \quad (7)$$

It is easily verified in the different cases that the complex integral represents the explicit time function on the left hand side.

Evaluating complex integrals of the type (1) is a frequently occurring task in the integrational calculus. For that purpose one has great use of the following *theorem* due to *Jordan* (Cf. Appendix 2):

Let $\Phi(p)$ be a function integrable in the region Re $\{p\} \geq \alpha$ along any smooth curve, when $|p| > c$, and such that $|\Phi(p)$ tends uniformly to zero when $|p| \to \infty$. Let C denote a half-circle with radius ϱ and centre at $(\alpha, 0)$ in the same region. Then

$$\lim_{\varrho=\infty} \int_C \Phi(p) \cdot e^{pt} \cdot dp = 0 \ldots\ldots\ldots\ldots\ldots \quad (8)$$

for all negative values of t.

Naturally a corresponding lemma subsists for positive values of t and an infinite half-circle convex to the left if $\Phi(p)$ fulfils the same conditions in the region $Re\{p\} \leq \alpha$.

The equality (8) is easily seen to be valid also in the case where the integration is extended over only a part of the half-circle C.

Now, the general method of treating an integral of the type

$$I = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(p) \cdot \frac{e^{pt}}{p} \cdot dp$$

is as follows:

According to a general proposition already mentioned for the class of functions $f(t)$ in question, the function $\frac{F(p)}{p}$ is holomorphical in the region $Re\{p\} > \alpha$, and there it also fulfils the conditions for Jordan's lemma. Another fundamental theorem, due to Cauchy, permits us to change the path L of integration into an infinite half-circle convex to the right. Then Jordan's lemma states I to be zero for all negative values of t. Of course we could know this fact *à priori*, but our reasoning has given us a sufficient condition for an integral I (t) to have this property.

The calculation of $I(t)$ when $t > 0$ necessitates a more complete knowledge of the analytic properties of $F(p)$ on the left hand side of the line $Re\{p\}=\alpha$. In most cases $\frac{F(p)}{p}$ has such properties that Jordan's lemma is applicable also in this half-plane. Then we may complete the line of integration by an infinite half-circle convex to the left, thus getting a contour C, inside which the integrand function possesses certain singularities as poles and branch-points; we assume the latter to be conveniently connected by cuts making the function uniform inside C. In accordance with Cauchy's theorem the integration along L may be replaced by integrations around the poles (residue calculus) and along the cuts. In a following chapter this procedure shall be abundantly exemplified.

2. The notion of integrator. Processes of differentiation and integration.

In the integral theorem to each function $f(t)$ another function $F(p)$ of the complex variable p is associated in a unique way. Obviously this function $F(p)$ is characteristic of and can be used to represent $f(t)$. Hence the unit-function $H(t)$, e. g., is represented by $F(p)=1$.

Now we are going to introduce a simplified notation for the integral expression (1), writing

$$f(t)=F(p)\cdot H(t). \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

Here the expression $F(p)$ symbolizes a definite operation of integration that is to be performed on the function $H(t)$. For that reason we may call $F(p)$ an *integrator*. As $f(t)$ is obtained by the application of $F(p)$ to the simplest and most fundamental function $H(t)$, we also will call $F(p)$ the *basic integrator* of $f(t)$. Of course, any function of class 1 possesses a uniquely defined basic integrator given by formula (2), but also other functions may be given by their basic integrators.

More generally, we shall call a function $F(p)$ of the complex variable p a basic integrator, when the corresponding integral (1) converges and remains stationary for all values of σ sufficiently large. Hence, by definition, the path L of integration is prescribed to be situated on the right of all singularities of $\frac{F(p)}{p}$.

Later in this chapter we shall take under discussion the interpretation of an integrational expression in some cases where the integral does not exist.

By the symbolic notation

$$\Phi(p)\cdot f(t)$$

which should be read: »integrator $\Phi(p)$ upon $f(t)$«, we understand the function $[\Phi(p)\cdot F(p)]\cdot H(t)$, if it exists.

From the definition of a basic integrator we immediately conclude the validity of certain arithmetic rules. Hence we have e. g.

$$\Phi(p)\cdot[k\cdot f(t)]=k\cdot\Phi(p)\cdot f(t),$$

$$\Phi(p)\cdot[f_1(t)+f_2(t)]=\Phi(p)\cdot f_1(t)+\Phi(p)\cdot f_2(t),$$

$$[\Phi_1(p)+\Phi_2(p)]\cdot f(t)=\Phi_1(p)\cdot f(t)+\Phi_2(p)\cdot f(t)$$

$$[\Phi_1(p)\cdot\Phi_2(p)]\cdot f(t)=\Phi_1(p)\cdot[\Phi_2(p)\cdot f(t)]=\Phi_2(p)\cdot[\Phi_1(p)\cdot f(t)]$$

$$[\Phi_1(p)\cdot\Phi_2(p)\cdot\Phi_3(p)]\cdot f(t)=$$

$$\Phi_1(p)\cdot\Phi_2(p)\cdot[\Phi_3(p)\cdot f(t)]=\Phi_1(p)\cdot[\Phi_2(p)\cdot\Phi_3(p)\cdot f(t)]$$

More precisely, we find the so-called distributive, commutative, and associative laws for processes of addition and multiplication to subsist in the calculus with basic integrators.

When applying different integrators $\Phi(p)$ to a given function $f(t)$ one obtains an unlimited number of new functions, which generally belong to other classes than the original one. Obviously, when $\Phi(p)$ is itself a basic integrator, the operation is always possible. When $\Phi(p)$ is not a basic integrator (e. g. $\Phi(p)=$ a polynomial in p), $\Phi(p)\cdot F(p)$ may be a basic integrator or not, according to the case.

The properties of the integrator function $F(p)$ depend of course wholly upon the *primary function* $f(t)$. To get some knowledge of that relationship we will perform integrations by parts in the definition formula (2). As $\sigma>\alpha$ we find:

$$\left|\lim_{\tau=\infty}\left[-\frac{f(\tau)}{p}\cdot e^{-p\tau}\right]\right|\leq\lim_{\tau=\infty}\left|\frac{1}{|p|}\cdot e^{-(\sigma-\alpha)\cdot\tau}\right|=0$$

and the first step of integration gives the result:

$$F(p)=f(0)+\int_0^\infty f'(\tau)\cdot e^{-p\tau}\cdot d\tau\text{. . . . .}\quad(14)$$

If there are exponentially limited derivatives of higher orders, the same procedure can be repeated one or more times. Thus, when $f(t)$ is of class n at least, one finds the following expression for $F(p)$:

$$F(p)=f(0)+\frac{f'(0)}{p}+\frac{f''(0)}{p^2}+\cdots+\frac{f^{(n-1)}(0)}{p^{n-1}}+\frac{1}{p^{n-1}}\int_0^\infty f^{(n)}(\tau)\cdot e^{-p\tau}\cdot d\tau \quad (15)$$

The integral appearing in the last term, which we may call the rest-term, is obviously uniformly limited in the domain

$$Re\{p\} \geq \sigma_0 > \alpha.$$

If $f(t)$ belongs to the class of order infinity, n may be any positive integer, and the rest-term will approach zero when $n \rightarrow \infty$, if $|p| > 1$. Hence, for such a function the following expansion holds:

$$F(p)=f(0)+\frac{f'(0)}{p}+\frac{f''(0)}{p^2}+\cdots+\frac{f^{(n)}(0)}{p^n}+\cdots \text{ in inf.} \quad (16)$$

Here the right hand side constitutes a power series in $\frac{1}{p}$ that, as we know, converges for some finite value of p. Outside any circle, concentric with and including the circle of convergence, the power series converges uniformly, and $F(p)$ is there uniformly limited. Thus we have the general theorem:

The basic integrator of a function $f(t)$ belonging to the class of order infinity is regular at the point at infinity.

The properties of $F(p)$ just stated make it possible, for $t>0$, to complete the path of integration L by an infinite half-circle, convex to the left, and then to perform term by term the complex integral:

$$I=\frac{1}{2\pi i}\int_{C+L}\left[f(0)+\frac{f'(0)}{p}+\frac{f''(0)}{p^2}+\cdots\right]\cdot\frac{e^{pt}}{p}\cdot dp$$

Now, the residue of the typical term $f^{(n)}(0)\cdot\frac{e^{pt}}{p^{n+1}}$ is $f^{(n)}(0)\cdot\frac{t^n}{\underline{|n}}$, and thus we find

$$f(t)=\left[f(0)+\frac{f'(0)}{\underline{|1}}\cdot t+\frac{f''(0)}{\underline{|2}}\cdot t^2+\cdots\right]\cdot H(t), \quad \ldots\ldots \quad (17)$$

which is the Taylor-expansion for $f(t)$. Any function of the class of order infinity may be represented in this way by a series, uniformly convergent in any finite interval of positive values of t.

As a special result we get from these considerations the important practical integrator rule:

$$\frac{1}{p^n}\cdot H(t)=\frac{t^n}{\underline{|n}}\cdot H(t) \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

for the present proved only for positive integers n.

In the preceding pages we have seen how a function $f(t)$ in a convertibly unique way is represented by its basic integrator $F(p)$. Now the question arises how the basic integrators of the derivatives $f'(t)$, $f''(t)$...., if they exist, are related to $F(p)$. The formula (14), or more generally (15), will give us some information in that respect.

If the function $f(t)$ is of class $(n+1)$ at least, $f^n(t)$ has a derivative, piecewise continuous and exponentially limited; hence the integral theorem is applicable to $f^n(t)$ and the basic integrator of this function will be

$$F_n(p)=p\cdot\int_0^\infty f^{(n)}(\tau)\cdot e^{-p\tau}\cdot d\tau \quad\ldots\ldots\ldots\ldots \quad (19)$$

From the formula (15) we obtain the expression desired for $F_n(p)$:

$$F_n(p)=p^n\cdot\left[F(p)-f(0)-\frac{f'(0)}{p}-\frac{f''(0)}{p^2}-\ldots-\frac{f^{(n-1)}(0)}{p^{n-1}}\right] \quad .. \quad (20)$$

or

$$F_n(p)=p^n\cdot F(p)-[f(0)\cdot p^n+f'(0)\cdot p^{n-1}+\cdots+f^{(n-1)}(0)\cdot p] \quad .. \quad (20a)$$

The last constituent of the right hand side, which we may call *the principal part* of $p^n F(p)$, has a convergence-producing task; by subtracting it from $p^n\cdot F(p)$ a convergent integral is obtained, which of course must be the case as this expression necessarily represents a basic integrator.

From (20) results how a linear differential expression with constant coefficients can be algebraized by the passage to the corresponding integrators.

On account of (20) we can pronounce the following *proposition:*

To apply to the function $f(t)$ the derivative symbol $\frac{d}{dt}$ means the same as multiplying its basic integrator $F(p)$ by p, provided that the initial value $f(0)$ is zero. To apply the integrative symbol $\int_0^t dt$ means, without restrictions, the same as multiplying the basic integrator by $\frac{1}{p}\cdot$

The latter part of the proposition follows immediately from the former one, if we note that $\varphi(t)=\int_0^t f(\tau)\,d\tau$ is a function with the properties: $\varphi(0)=0$, $\varphi'(t)=f(t)$.

Using our mode of writing, when $f(0)=0$, we have the equalities

$$\frac{d}{dt}\,f(t)=\frac{d}{dt}\,F(p)\cdot H(t)=p\cdot F(p)\cdot H(t)=p\cdot f(t), \quad \ldots . \qquad 1)$$

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$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} p\cdot F(p)\cdot\frac{e^{pt}}{p}\cdot dp,$$

thus a factor p before $F(p)$. But in many cases the new integral does not converge, and then the process of differentiation cannot be performed in this way. For this a uniform convergence of the new integral is required. It is, however, possible to modify the original integral in such a way that the procedure is correct, provided that $F(p)$ satisfies certain conditions. Leaving aside the question of the most general form of these conditions, we will make some assumptions relative to $F(p)$ that are usually fulfilled in the practice.

We assume $\frac{F(p)}{p}$ to be holomorphical in the region to the right of the straight lines $p=\alpha+r\cdot e^{i\varphi}$ and $p=\alpha+r\cdot e^{-i\varphi}$, where $\pi \geq \varphi > \frac{\pi}{2}$, and such that $\left|\frac{F(p)}{p}\right|$ tends uniformly to zero when $p \to \infty$ in this region or in any region contained by it (Cf. fig. 2). Such a function or basic integrator is said, for brevity, to satisfy the condition *A*.

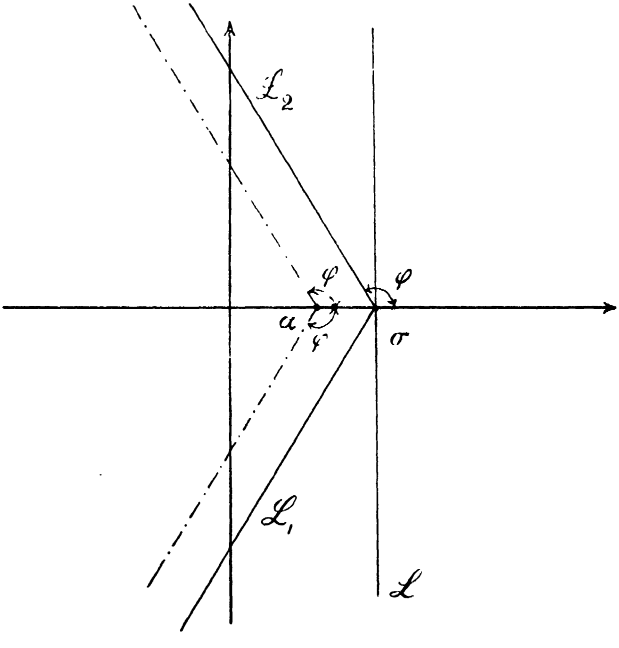

Fig. 2.

As before we conclude that the function $f(t)=F(p)\cdot H(t)$ is zero identically when t is negative. When t is positive we may, on account of the properties stated about $F(p)$, complete the line L of integration by parts of an infinite half-circle, convex to the left, and with its centre at $(\sigma, 0)$, and then, without traversing any singularities, change the path of integration into two straight lines L_1 and L_2, as shown by the figure. The integral along L_1 and L_2 we will call the *modified integral.* Now, this integral is easily proved to be uniformly convergent for all $t \geq t_0$ ($t_0 =$ an arbitrarily small positive number)*, and this is also the case with any integral of the type

$$\frac{1}{2\pi i}\cdot\int\limits_{L_1\,L_2} Q(p)\cdot F(p)\cdot\frac{e^{pt}}{p}\cdot dp, \quad \ldots\ldots\ldots\ldots\ldots \quad (\beta)$$

where $Q(p)$ denotes a function that in the region in question increases at most as a power of p when $|p| \to \infty$.

From this we conclude that, when an integrator $F(p)$ satisfies the condition A, the corresponding primary function $f(t)$ *for positive values of* t has an unlimited number of derivatives, and that the symbol $\frac{d}{dt}$ before $f(t)$ means the same as the factor p before $F(p)$ in the modified integral and conversely. One should observe that the new functions $p\cdot F(p)$, $p^2\cdot F(p)$ etc. as a rule do not satisfy the condition A or have the character of basic integrators.

Thus it proves possible to establish for the specified class of functions $F(p)$ such an equivalence between the derivative symbol $\frac{d}{dt}$ and the integrator symbol p that a well defined operational meaning can be given to the expression $\Phi(p)\cdot f(t)$ for positive values of t, also in cases where $\Phi(p)\cdot F(p)$ is not a basic integrator. When the integral (1) does not converge, we will always interpret this expression by the modified integral, provided that $F(p)$ and $\Phi(p)$ fulfil the conditions stated.

At this point, however, we must emphasize that the commutative law, which is valid for basic integrators, does not always subsist for this more general class of integrators. This may be exemplified in a simple case. The expression

$$p^{\frac{5}{2}}\cdot H(t)$$

means, in accordance with the definition, if $t > 0$:

$$p^{\frac{5}{2}}\cdot H(t) = \frac{1}{2\pi i}\int\limits_{L_1 L_2} p^{\frac{5}{2}}\cdot\frac{e^{pt}}{p}dp = \frac{d^2}{dt^2}\cdot\frac{1}{2\pi i}\cdot\int\limits_{L_1 L_2} p^{\frac{1}{2}}\cdot\frac{e^{pt}}{p}dp = \frac{d^2}{dt^2}\cdot p^{\frac{1}{2}}\cdot H(t).$$

---

* Cf. Appendix 4.

Thus one has the right to make the factorial decomposition $p^{\frac{5}{2}}=p^2\cdot p^{\frac{1}{2}}$ and then to replace p^2 by the symbol $\frac{d^2}{dt^2}$. Quite as well, however, one could try to proceed in another way, viz

$$p^{\frac{5}{2}}\cdot H(t)=p^{\frac{1}{2}}\cdot p^2\cdot H(t)=p^{\frac{1}{2}}\cdot\frac{d^2}{dt^2}\cdot H(t)=0.$$

Obviously this procedure is wrong as it violates the integral of definition. Hence the commutative law cannot be applied to the product $(p^2\cdot p^{\frac{1}{2}})\cdot H(t)$.

As a general rule all operations of differentiation ought to be performed last in the suit of operations occurring. Indeed, by differentiations certain constituents of the time function can disappear that would perhaps have some influence by another mode of proceeding. In any case the correctness of the procedure can be checked by an examination of the modified integral.

Making use of our generalized class of integrators we will now evaluate $f^{(n)}(t)$ according to (20 a)

$$f^{(n)}(t)=\{p^n\cdot F(p)-[f(0)\cdot p^n+f'(0)\cdot p^{n-1}+\cdots+f^{(n-1)}(0)\cdot p]\}\cdot H(t) \tag{22}$$

The modified integral* gives, when $t>0$,

$$\begin{aligned} f^{(n)}(t) &= \frac{1}{2\pi i}\int_{L_1L_2} p^n\cdot F(p)\cdot\frac{e^{pt}}{p}\cdot dp \\ &-\frac{1}{2\pi i}\int_{L_1L_3}[f(0)\cdot p^n+f'(0)\cdot p^{n-1}+\cdots+f^{(n-1)}(0)\cdot p]\cdot\frac{e^{pt}}{p}\cdot dp. \end{aligned} \tag{23}$$

Now, for the path of integration in question the following general equality holds:

$$J_n\equiv\int_{L_1L_3} p^n\cdot e^{pt}\,dp=0 \;\ldots\ldots\ldots\ldots\ldots\ldots \tag{24}$$

and hence, if $t>0$,

$$f^{(n)}(t)=\frac{1}{2\pi i}\int_{L_1L_2} p^n\cdot F(p)\cdot\frac{e^{pt}}{p}\cdot dp \;\ldots\ldots\ldots\ldots \tag{25}$$

---

* $F_n\ (p)$ is supposed to satisfy the condition A.

The disappearance of the last part of the right hand side of (22) for positive values of t will of course also occur if we interpret the symbol p as $\frac{d}{dt}$.

Instead of employing the modified integral to interpret an integrational expression $F(p) \cdot H(t)$ when $F(p)$ is not a basic integrator, one can make use of the practical rule of simply *suppressing the principal parts* of the expressions, which are often easily isolated, and then retaining the original path of integration L. In fact, that is what was done, when in (20a) the parenthesis [ ] was subtracted from $p^n \cdot F(p)$.

The procedure may be elucidated by a simple example. From the foregoing we have the rules

$$\frac{p^2}{p^2+\omega^2} \cdot H(t) = \cos \omega t \cdot H(t); \ \frac{\omega p}{p^2+\omega^2} \cdot H(t) = \sin \omega t \cdot H(t). \ \ldots \quad (26)$$

Suppose we have to interpret the expression:

$$f(t) = \frac{ap^4 + b \cdot p^3}{p^2+\omega^2} \cdot H(t) \ \ldots\ldots\ldots\ldots\ldots\ldots \quad (27)$$

The integral of definition

$$I = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{ap^4 + bp^3}{p^2+\omega^2} \cdot \frac{e^{pt}}{p} \cdot dp \ \ldots\ldots\ldots\ldots\ldots \quad (28)$$

has no meaning in this form. By a simple process of division we obtain

$$\frac{ap^4 + bp^3}{p^2+\omega^2} = ap^2 + bp - a\omega^2 \cdot \frac{p^2}{p^2+\omega^2} - b\omega \cdot \frac{\omega p}{p^2+\omega^2}. \ \ldots\ldots \quad (29)$$

If here we suppress the principal part and introduce the remaining constituents in (28), we get two convergent integrals, which, in accordance with (26), can be at once identified with

$$-a \cdot \omega^2 \cdot \cos \omega t \cdot H(t) \text{ and } -b\omega \cdot \sin \omega t \cdot H(t).$$

Hence the function in question is

$$f(t) = (-a\omega^2 \cdot \cos \omega t - b\omega \cdot \sin \omega t) \cdot H(t) \ \ldots\ldots\ldots \quad (30)$$

A simpler proceeding is as follows:

$$f(t)=a\cdot p^2\frac{p^2}{p^2+\omega^2}\cdot H(t)+\frac{b}{\omega}\cdot p^2\frac{\omega p}{p^2+\omega^2}\cdot H(t)$$

$$=\left(a\cdot\frac{d^2}{dt^2}\cos\omega t+\frac{b}{\omega}\cdot\frac{d^2}{dt^2}\cdot\sin\omega t\right)\cdot H(t)\;\ldots\ldots\ldots\;(31)$$

$$=(-a\cdot\omega^2\cdot\cos\omega t-b\cdot\omega\cdot\sin\omega t)\cdot H(t).$$

At last we will determine the basic integrator of $f(t)$. This can be done in two ways. On one hand we obtain the integrator wanted from (29) after suppressing the principal part. Hence

$$F(p)=-\omega^2\cdot\frac{ap^2+bp}{p^2+\omega^2}\;\ldots\ldots\ldots\ldots\ldots\ldots\;(32)$$

On the other hand, according to (30), $F(p)$ is the sum of the basic integrators of $-a\omega^2\cdot\cos\omega t\cdot H(t)$ and $-b\omega\cdot\sin\omega t\cdot H(t)$. Thus, from (26) we get back the equality (32).

---

3. A linear integral operator and its properties of composition.

Let $f(t)$ be a function of class 1 at least. We start from the identity

$$f(t)=f(0)\cdot H(t)+\int_0^t f'(\tau)\cdot d\tau,$$

which may also be written

$$f(t)=f(0)\cdot H(t)+\int_0^t f'(\tau)\cdot H(t-\tau)\cdot d\tau,\;\ldots\ldots\ldots\;(33)$$

and obtain, taking $t-\tau$ as a new variable of integration,

$$f(t)=f(0)\cdot H(t)+\int_0^t f'(t-\tau)\cdot H(\tau)\,d\tau.\;\ldots\ldots\;(34)$$

The last identity can be written in a symbolic manner

$$f(t)=[f(0)+\int_0^t f'(t-\tau)\cdot d\tau]\times H(t).\;\ldots\ldots\ldots\;(35)$$

Accordingly, with any function $f(t)$ we can associate an *integral operator* F:

$$F=[f(0)+\int_0^t f'(t-\tau)\cdot d\tau] \quad \ldots\ldots\ldots\ldots\ldots \quad (36)$$

and generally define a symbolic product:

$$F\times\varphi(t)=f(0)\cdot\varphi(t)+\int_0^t f'(t-\tau)\cdot\varphi(\tau)\cdot d\tau. \quad \ldots\ldots \quad (37)$$

Remark. The term »operator» is here reserved for such symbols as indicate a process of direct differentiation or integration to be performed on a function. The denomination »integrator» is employed for symbols denoting that a new function shall be derived from one already given by some well defined procedure of integration of a more indirect nature. In many cases the same expression can simultaneously symbolize an operator and an integrator.

The common arithmetical laws are applicable to the functional operator F just introduced. From its linear nature we at once conclude the validity of the first three rules given for integrators on page 12. To get the rules for symbolic multiplication we must examine the composition properties of the operator.

Let $f_1(t)$ and $f_2(t)$ be two functions with the corresponding operators

$$F_1=[f_1(0)+\int_0^t f'_1(t-\tau)\,d\tau] \text{ and } F_2=[f_2(0)+\int_0^t f'_2(t-\tau)\,d\tau].$$

We shall have to compare the results in the two cases of composition

$$F_1\times(F_2\times f(t)) \text{ and } F_2\times(F_1\times f(t)),$$

where $f(t)$ denotes any function. According to the definition we get

$$\varphi(t)=[f_1(0)+\int_0^t f'_1(t-\tau)\cdot d\tau]\times[f_2(0)\cdot f(t)+\int_0^t f'_2(t-\tau)\cdot f(\tau)\cdot d\tau]=$$

$$=f_1(0)\cdot f_2(0)\cdot f(t)+f_1(0)\cdot\int_0^t f'_2(t-\tau)\cdot f(\tau)\cdot d\tau+f_2(0)\cdot\int_0^t f_1'(t-\tau)\cdot f(\tau)\cdot d\tau$$

$$+\int_0^t d\tau \int_0^\tau f_1'(t-\tau)\cdot f_2'(\tau-\tau_1)\cdot f(\tau_1)\cdot d\tau_1.$$

The double integral can be transformed conformably to a method due to Lejeune-Dirichlet (Cf. fig. 3).

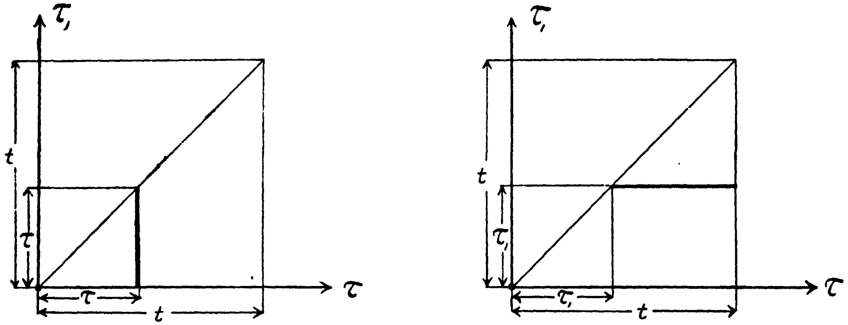

Fig. 3.

$$\int_0^t d\tau \int_0^\tau f_1'(t-\tau)\cdot f_2'(\tau-\tau_1)\cdot f(\tau_1)\cdot d\tau_1 =$$

$$=\int_0^t f(\tau_1)\cdot d\tau_1 \int_{\tau_1}^t f'_1(t-\tau)\cdot f'_2(\tau-\tau_1)\cdot d\tau \;\ldots\ldots\ldots\ldots \quad (38)$$

Hence, with the notation

$$f'_{12}(t-\tau)=\int_\tau^t f'_1(t-\xi)\cdot f'_2(\xi-\tau)\cdot d\xi, \;\ldots\ldots\ldots \quad (39)$$

and using a symbolic mode of writing, we obtain

$$F_1\times(F_2\times f(t))= \;\ldots\ldots\ldots\ldots\ldots\ldots \quad (40)$$

$$[f_1(0)\cdot f_2(0)+f_1(0)\cdot\int_0^t f'_2(t-\tau)\cdot d\tau+f_2(0)\cdot\int_0^t f'_1(t-\tau)\cdot d\tau+$$

$$+\int_0^t f'_{12}(t-\tau)\cdot d\tau]\times f(t).$$

It is easily seen that $f'_{12}(t,\tau)$ is really a function of the argument $(t-\tau)$; indeed, a simple change of variable of integration proves that a function of the type

$$F(x,y)=\int_x^y u(y-\xi)\cdot v(\xi-x)\cdot d\xi$$

satisfies the equality $F(x+a,y+a)=F(x,y)$ for any value of a. It is also simple to verify that such a function $F(x,y)$ is symmetric in u and v, i. e. remains unaltered when u and v change places.

Hence we have the identity

$$f'_{21}(t-\tau)=\int_\tau^t f'_2(t-\xi)\cdot f'_1(\xi-\tau)\cdot d\xi=f'_{12}(t-\tau). \quad \ldots . \quad (41)$$

Now, if we treat the second composed operator $F_2\times(F_1\times f(t))$ in a similar manner, we must, by symmetry, get the same result as in (40), if we let the indices 1 and 2 change places. But, on account of (41), (40) is invariant against such a procedure, and thus we may write:

$$F_1\times(F_2\times f(t))=F_2\times(F_1\times f(t))=(F_1\times F_2)\times f(t) \quad \ldots . \quad (42)$$

Accordingly, two integral operators F_1 and F_2, which operate consecutively on a given function, their order being immaterial, may be replaced by a new operator, which can be regarded as obtained from the former by a kind of symbolic multiplication in agreement with the principle:

$$F_1\times F_2=F_2\times F_1=$$

$$[f_1(0)+\int_0^t f'_1(t-\tau)\cdot d\tau]\times[f_2(0)+\int_0^t f'_2(t-\tau)\cdot d\tau]= \quad .. \quad (43)$$

$$[f_1(0)\cdot f_2(0)+f_1(0)\cdot\int_0^t f'_2(t-\tau)\cdot d\tau+f_2(0)\cdot\int_0^t f'_1(t-\tau)\cdot d\tau+$$

$$+\int_0^t f'_{12}(t-\tau)\cdot d\tau].$$

We have just found the commutative law to be valid for this product. The associative law of multiplication, expressed by the equalities

$$F_1\times F_2\times F_3=F_1\times(F_2\times F_3)=F_2\times(F_3\times F_1)=F_3\times(F_2\times F_1),\ldots \quad (44)$$

can easily be proved in a similar manner. In fact, repeating the procedure employed, we get:

$$\begin{gathered}F_1\times F_2\times F_3=\\
[f_1(0)+\int_0^t f'_1(t-\tau)\cdot d\tau]\times[f_2(0)+\int_0^t f'_2(t-\tau)\cdot d\tau]\times[f_3(0)+\\
+\int_0^t f'_3(t-\tau)\cdot d\tau]=[f_1(0)\cdot f_2(0)\cdot f_3(0)+\\
\qquad\qquad\qquad\qquad\qquad\qquad\ldots\ldots \quad (45)\\
+f_1(0)\cdot f_2(0)\cdot\int_0^t f'_3(t-\tau)\cdot d\tau+f_3(0)\cdot f_1(0)\cdot\int_0^t f'_2(t-\tau)\cdot d\tau+\\
+f_2(0)\cdot f_3(0)\cdot\int_0^t f'_1(t-\tau)\cdot d\tau+f_1(0)\cdot\int_0^t f'_{23}(t-\tau)\cdot d\tau+\\
+f_2(0)\cdot\int_0^t f'_{31}(t-\tau)\cdot d\tau+f_3(0)\cdot\int_0^t f'_{12}(t-\tau)\cdot d\tau+\int_0^t f'_{123}(t-\tau)\cdot d\tau].\end{gathered}$$

The meaning of $f'_{rs}(t-\tau)$ ($r, s=1, 2, 3,\ r\neq s$) follows from (41). The function $f_{123}'(t-\tau)$ can be defined e. g. by the integral

$$f'_{(12)3}(t-\tau)=\int_\tau^t f'_{12}(t-\xi)\cdot f'_3(\xi-\tau)\cdot d\xi \ \ldots\ldots\ldots \quad (46)$$

Using the method of Lejeune-Dirichlet one can at once verify that

$$f'_{123}=f'_{(12)3}=f'_{1(23)}=f'_{2(13)}=\text{etc.}\ \ldots\ldots\ldots\ldots \quad (47)$$

Thus the indices may be arbitrarily permuted and associated.

It is often more covenient to write composed integrators under the original form (36). Then, instead of (43) we obtain

$$F_1 \times F_2 = [f_1(0) \cdot f_2(0) + \int_0^t K_{12}(t-\tau) \cdot d\tau], \ldots\ldots\ldots \quad (48)$$

where

$$K_{12}(x) = f_1(0) \cdot f'_2(x) + f_2(0) \cdot f'_1(x) + f'_{12}(x) = K_{21}(x). \;\ldots\ldots \quad (49)$$

We call $K_{12}(t-\tau)$ the *integral kernel* of the operator $F_1 \times F_2$. To illustrate the general rule of forming the integral kernel we represent the expression of the product of four factors:

$$F_1 \times F_2 \times F_3 \times F_4 = [f_1(0) \cdot f_2(0) \cdot f_3(0) \cdot f_4(0) + \int_0^t K_{1234}(t-\tau) \cdot d\tau] \quad (50)$$

$$\begin{gathered} K_{1234}(x) = \\ f'_1(x) \cdot f_2(0) \cdot f_3(0) \cdot f_4(0) + f'_2(x) \cdot f_3(0) \cdot f_4(0) \cdot f_1(0) + \\ + f'_3(x) \cdot f_4(0) \cdot f_1(0) \cdot f_2(0) + f'_4(x) \cdot f_1(0) \cdot f_2(0) \cdot f_3(0) + \\ + f'_{12}(x) \cdot f_3(0) \cdot f_4(0) + f'_{13}(x) \cdot f_2(0) \cdot f_4(0) + f'_{14}(x) \cdot f_2(0) \cdot f_3(0) + \\ + f'_{23}(x) \cdot f_1(0) \cdot f_4(0) + f'_{24}(x) \cdot f_1(0) \cdot f_3(0) + f'_{34}(x) \cdot f_1(0) \cdot f_2(0) + \\ + f'_{123}(x) \cdot f_4(0) + f'_{234}(x) \cdot f_1(0) + f'_{341}(x) \cdot f_2(0) + f'_{412}(x) \cdot f_3(0) + \\ + f'_{1234}(x). \end{gathered}$$

As wee see this expression will be considerably simplified if some of the initial values $f_\nu(0)$ are zero. When they all vanish, we simply get:

$$F_1 \times F_2 \times F_3 \times F_4 = [\int_0^t f'_{1234}(t-\tau) \cdot d\tau] \;\ldots\ldots\ldots \quad (50\text{ a})$$

Of special importance is the case where all the operational factors are equal, that is when the product is a power of the symbol F. Then we conveniently use simple indices for the functions f' and K. The original function being $f(t)$, and accordingly

$$F = [f(0) + \int_0^t f'(t-\tau) \cdot d\tau],$$

we obtain the following general expression for an integer power of F:

$$\left.\begin{aligned} F^n&=\Big[f(0)^n+\int_0^t K_n(t-\tau)\cdot d\tau\Big], \\ \text{where}\quad & \\ K_n(x)&=\sum_{\nu=1}^{n}\binom{n}{\nu}\cdot f(0)^{n-\nu}\cdot f'_\nu(x), \\ f'_n(t-\tau)&=\int_\tau^t f'_{n-1}(t-\xi)\cdot f'(\xi-\tau)\cdot d\xi;\ f'_1(x)=f'(x). \end{aligned}\right\} \quad \ldots (51)$$

The proof is readily carried out from n to $n+1$ in accordance with the mode of proceeding just employed in the general case.

4*. The equivalence between integrators and integral operators. The inverse operations.

According to the foregoing we can associate with any function $f(t)$ two operational symbols, the integrator $F(p)$ and the integral operator F, which, if applied to other functions, give rise to new ones. The definitions were:

$$F(p)=f(0)+\int_0^\infty f'(\tau)\cdot e^{-p\tau}d\tau;\quad F=\Big[f(0)+\int_0^t f'(t-\tau)\cdot d\tau\Big]. .\ (52)$$

Operating upon the unit-function, $H(t)$, both give back the original function $f(t)$. Operating on an arbitrary function $\varphi(t)$, they will both give the same new function, that is:

$$F(p)\cdot\varphi(t)=F\times\varphi(t).\ \ldots\ldots\ldots\ldots\ (53)$$

To prove this identity we will make use of an auxiliary theorem, the correctness of which immediately follows from the definition of an integrator, viz

$$e^{-p\tau}\cdot\Phi(p)\cdot H(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Phi(p)\cdot\frac{e^{p(t-\tau)}}{p}\cdot dp=[\Phi(p)\cdot H(t)]_{t-\tau}.\ (54)$$

At first we transform the right hand side of (53), utilizing $\varphi(t)$'s property of vanishing for negative values of t.

[*] Cf. Appendix 7.

$$F\times\varphi(t)=f(0)\cdot\varphi(t)+\int_0^t f'(\tau)\cdot\varphi(t-\tau)\cdot d\tau$$

$$=f(0)\cdot\varphi(t)+\int_0^\infty f'(t)\cdot\varphi(t-\tau)\cdot d\tau$$

$$=f(0)\cdot\varphi(t)+\int_0^\infty f'(\tau)\cdot[\Phi(p)H(t)]_{t-\tau}\cdot d\tau.$$

Hence, on account of (54) and the integral theorem (Cf. Appendix 1),

$$F\times\varphi(t)=f(0)\cdot\Phi(p)\cdot H(t)+\int_0^\infty f'(\tau)\cdot e^{-p\tau}\cdot\Phi(p)\cdot H(t)\cdot d\tau$$

$$=[f(0)+\int_0^\infty f'(\tau)\cdot e^{-p\tau}\cdot d\tau]\cdot\Phi(p)\cdot H(t),$$

and at last, according to (52):

$$F\times\varphi(t)=F(p)\cdot\Phi(p)\cdot H(t)=F(p)\cdot\varphi(t).$$

We also know the symbols $F(p)$ and F to obey the common arithmetic laws of addition and multiplication, and therefore we can regard them as equivalent when simultaneously appearing in expressions obtained by these operations of calculus. This fact is of great practical importance for the following reason:

The employment of integrators for solving differential equations will present the unknown function under the form

$$f(t, x, y, z\ldots)=F(p;\ x, y, z\ldots)\cdot\varphi(t)\ldots\ldots\ldots \quad (55)$$

To find the explicit function of time one can of course always fall back upon the definition-formula (1) and try to evaluate the complex integral. In many cases, however, the function $F(p;\ x, y, z\ldots)$ can be broken up into terms and factors in different ways, and these simpler constituents are often of similar types from one case to another. Thus, having once derived the integral operators, corresponding to a number of such typical integrators, one can, on account of the equivalence just mentioned, make use of these results and in this way often simplify the interpretation of composed integrational expressions.

Assume e. g. we want to find $x(t)$ when

$$x(t)=F_1(p)\cdot F_2(p)\cdot F_3(p)\cdot F_4(p)\cdot\varphi(t), \quad\ldots\ldots\ldots\quad (56)$$

and that we have already obtained the equalities

$$F_1(p)\cdot H(t)=F_1\times H(t), \quad F_3(p)\cdot H(t)=F_3\times H(t).$$

Then we rewrite $x(t)$ in the form

$$x(t)=F_2(p)\cdot F_4(p)\cdot F_1\times F_3\times\varphi(t), \quad\ldots\ldots\ldots\ldots\quad (57)$$

and have only to evaluate the complex integral

$$u(t)=\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}F_2(p)\cdot F_4(p)\cdot\frac{e^{pt}}{p}\cdot dp.$$

This brings us the operator

$$U=[u(0)+\int\limits_0^t u'(t-\tau)\,d\tau].$$

Hence, the final computation of $x(t)$,

$$x(t)=U\times F_1\times F_3\times\varphi(t), \quad\ldots\ldots\ldots\ldots\ldots\quad (58)$$

according to the rules of composition of integral operators, is lead back to pure quadratures.

The method of calculus described above is not always the most convenient from a practical point of view. It is often better to retain the form (55) of the integrational expression and sometimes even to introduce $\Phi(p)\cdot H(t)$ instead of $\varphi(t)$. This is especially the case, when $F(p)\cdot\Phi(p)$ is of such a form (e. g. is a rational function) that one of the so-called expansion theorems can be used. (See later in Chapter 7.)

The relationship between a basic integrator $F(p)$ and the corresponding operator F can be elucidated from another side by a study of the *inverse* operations.

Proposed that $\varphi(t)$ is any given function with the basic integrator $\Phi(p)$, on applying $F(p)$ to $\varphi(t)$ we obtain a new function, say $\psi(t)$

$$\psi(t)=F(p)\cdot\varphi(t), \quad\ldots\ldots\ldots\ldots\ldots\ldots\quad (59)$$

the certainly existing basic integrator of which is

$$\Psi(p)=F(p)\cdot\Phi(p). \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (60)$$

The function $\psi(t)$ can also be defined by means of the integral operator F:

$$\psi(t)=F\times\varphi(t)=f(0)\cdot\varphi(t)+\int_0^t f'(t-\tau)\cdot\varphi(\tau)\cdot d\tau \ldots\ldots \quad (61)$$

Our examination now refers to the opposite or inverse operations, which bring us back from $\psi(t)$ to $\varphi(t)$. From (60) we get

$$\Phi(p)=\frac{1}{F(p)}\cdot\Psi(p)\equiv U(p)\cdot\Psi(p). \quad\ldots\ldots\ldots \quad (62)$$

Therefore

$$\varphi(t)=\frac{1}{F(p)}\cdot\psi(t)=U(p)\cdot\psi(t). \quad\ldots\ldots\ldots\ldots \quad (63)$$

Provided that $U(p)$ is itself a basic integrator, employing its corresponding operator U, we may write $\varphi(t)$ under the form

$$\varphi(t)=U\times\psi(t)=u(0)\cdot\psi(t)+\int_0^t u'(t-\tau)\cdot\psi(\tau)\,d\tau \ldots\ldots \quad (64)$$

Now, the equations (64) and (61) being reversed to one another, we ought to establish a functional relationship between $f'(t-\tau)$ and $u'(t-\tau)$ by inserting the expression (61) of $\psi(t)$ into (64). By this means we get the identity:

$$\varphi(t)=u(0)\cdot f(0)\cdot\varphi(t)+u(0)\cdot\int_0^t f'(t-\tau)\cdot\varphi(\tau)\,d\tau+$$

$$+\int_0^t u'(t-\tau)\,d\tau\cdot\left[f(0)\cdot\varphi(\tau)+\int_0^\tau f'(\tau-\tau_1)\cdot\varphi(\tau_1)\cdot d\tau_1\right]. \quad\ldots \quad (65)$$

The double integral may be transformed in a well-known way, and thus we obtain:

$$[1-f(0)\cdot u(0)]\cdot\varphi(t)= \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (66)$$

$$\int_0^t \varphi(\tau)\cdot d\tau\cdot\left[u(0)\cdot f'(t-\tau)+f(0)\cdot u'(t-\tau)+\int_\tau^t u'(t-\xi)\cdot f'(\xi-\tau)\cdot d\xi\right].$$

Now, if this homogeneous integral equation shall be satisfied for arbitrary functions $\varphi(t)$, the following two conditions must be fulfilled:

$$\left.\begin{aligned} &f(0)\cdot u(0)=1; \\ &u(0)\cdot f'(t-\tau)+f(0)\cdot u'(t-\tau)+\int_{\tau}^{t} f'(t-\xi)\cdot u'(\xi-\tau)\cdot d\xi=0 \end{aligned}\right\} \quad ..(67)$$

These can be combined to one, if we introduce the notations

$$f'(t-\tau)=f(0)\cdot f^{*}(t-\tau),\quad u'(t-\tau)=u(0)\cdot u^{*}(t-\tau). \;\ldots\ldots\; (68)$$

Then, instead of (67) we get the only condition

$$f^{*}(t-\tau)+u^{*}(t-\tau)+\int_{\tau}^{t} f^{*}(t-\xi)\cdot u^{*}(\xi-\tau)\cdot d\xi=0. \;\ldots\ldots\; (69)$$

We will call U the *inverse* integral operator to F. As for any function $\varphi(t)$ we have $\varphi(t)=U\times F\times\varphi(t)$, the relation $U\times F=1$ subsists, and instead of U we may conveniently use the notation

$$U=F^{-1}. \;\ldots\ldots\ldots\ldots\ldots\ldots\; (70)$$

According to (67) a necessary condition for the operator F to possess an inverse operator is that $f(0)\neq 0$. Then also $u(0)\neq 0$, and the functional relation (69) subsists between the functions $u^{*}(t-\tau)$ and $f^{*}(t-\tau)$. Two functions, linked together by this relationship are said to be *reciprocal* to each other. The equation (69) is another mode of writing the relation $F(p)\cdot U(p)=1$ between the basic integrators or the relation $F\times U=1$ between the integral operators. It constitutes an *integral equation* of *Volterra's type.* If we regard $f^{*}(y)$ as the given function and $u^{*}(y)$ as the function to be determined, we can e. g. rewrite the equation under the standard form

$$u^{*}(y)+\int_{0}^{y} f^{*}(y-x)\cdot u^{*}(x)\,dx=-f^{*}(y), \;\ldots\ldots\; (69\text{ a})$$

and then obtain the unique solution by forming the *resolvent* according to a wellknown general method.

More interest is offered by the expression for the inverse operator F^{-1}, which is easily derived in a similar way. One will find, by solving the Volterra equation (61), that

$$F^{-1}=[f(0)^{-1}+\int_{0}^{t}\mathrm{F}(t-\tau)\,d\tau], \qquad (71)$$

where

$$\mathrm{F}(x)=\sum_{\nu=1}^{\infty}(-1)^{\nu}\cdot\frac{f'_{\nu}(x)}{f(0)^{\nu+1}}. \qquad (71\text{ a})$$

The functions $f'_{\nu}(x)$ are obtained from (51).

As a simple example of the inverse operations we take the integrational equation:

$$F(p)\cdot\varphi(t)=\frac{p^2}{p^2+\omega^2}\cdot\varphi(t)=\psi(t) \qquad (\alpha)$$

Here the operator is

$$F=[1-\omega\int_{0}^{t}\sin\omega(t-\tau)\cdot d\tau].$$

Regarded as the unknown function, $\varphi(t)$ is determined by the integral equation

$$\varphi(t)-\omega\cdot\int_{0}^{t}\sin\omega(t-\tau)\cdot\varphi(\tau)\cdot d\tau=\psi(t) \qquad (\beta)$$

On the other side $\varphi(t)$ can also be obtained from the equation (α). Hence

$$\varphi(t)=\left(1+\frac{\omega^2}{p^2}\right)\cdot\psi(t).$$

Here the right hand side can at once be interpreted, and we get

$$\varphi(t)=\psi(t)+\omega^2\int_{0}^{t}d\tau\int_{0}^{\tau}\psi(\tau_1)\cdot d\tau_1$$

or

$$\varphi(t)=\psi(t)+\omega^2\cdot\int_{0}^{t}(t-\tau)\cdot\psi(\tau)\cdot d\tau. \qquad (\gamma)$$

Therefore, the inverse integral operator of F will be

$$F^{-1}=[1+\omega^2\cdot\int_0^t(t-\tau)\cdot d\tau].$$

It is easy to check that $F\times F^{-1}=1$, e. g. by inserting the expression for $\psi(t)$ according to (β) into (γ); a pure identity is then obtained.

Our considerations furnish us with a general method of solving a Volterra equation of the special type:

$$\varphi(t)+\int_0^t K(t-\tau)\,\varphi(t)\cdot d\tau=\psi(t). \quad \ldots\ldots\ldots \quad (\delta)$$

This becomes identical with (61) if we put $f(0)=1$ and $K(x)=f'(x)$. On account of the first of the formulas (52) the corresponding basic integrator is

$$F(p)=1+\int_0^\infty K(\tau)\cdot e^{-p\tau}d\tau. \quad \ldots\ldots\ldots\ldots\ldots \quad (\varepsilon)$$

If this integral is comparatively simply evaluated*, it is in many cases practical to present the solution of the integral equation (δ) in the form of the integrational expression

$$\varphi(t)=\frac{1}{F(p)}\cdot\psi(t). \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\eta)$$

In our example we had $K(\tau)=-\omega\sin\omega\tau$; hence

$$F(p)=1-\omega\cdot\int_0^\infty\sin\omega\tau\cdot e^{-p\tau}d\tau=\frac{p^2}{p^2+\omega^2},$$

and the equation (η) gets a very simple form.

---

* It may happen that $K(\tau)$ can be approximated by a suitable aggregate of polynomials and exponentials.

5. A table of some important integrational expressions and rules of calculus. Applications of the general methods of the foregoing chapters.

In the table below we present a collection of some simple integrational formulas that are of frequent use in the practice. This collection will be subsequently completed in the following.

1. $$\frac{1}{p^n}\cdot H(t)=\frac{t^n}{\underline{|n}}\cdot H(t)$$

2. $$\frac{p}{p-\alpha}\cdot H(t)=e^{\alpha t}\cdot H(t)$$

3. $$\frac{p}{(p-\alpha)^n}\cdot H(t)=\frac{t^{n-1}}{\underline{|n-1}}\cdot e^{\alpha t}\cdot H(t)$$

4. $$\frac{1}{p-\alpha}\cdot H(t)=\frac{1}{\alpha}(e^{\alpha t}-1)\cdot H(t)$$

5. $$\frac{p^2}{p^2+\omega^2}\cdot H(t)=\cos\omega t\cdot H(t)$$

6. $$\frac{\omega p}{p^2+\omega^2}\cdot H(t)=\sin\omega t\cdot H(t)$$

7. $$p\cdot\frac{(p+\beta)\cdot\sin\varphi+\omega\cdot\cos\varphi}{(p+\beta)^2+\omega^2}\cdot H(t)=e^{-\beta t}\cdot\sin(\omega t+\varphi)\cdot H(t)$$

8. $$\Phi(p+a)\cdot f(t)=e^{-at}\cdot\Phi(p)\cdot e^{at}\cdot f(t)$$

8 a. $$\Phi(p)\cdot f(t)=e^{at}\cdot\Phi(p+a)\cdot e^{-at}\cdot f(t)$$

9. $$e^{ap}\cdot f(t)=f(t+a)$$

10. If $F(p)\cdot H(t)=f(t)$, then $F(\lambda p)\cdot H(t)=f\left(\frac{t}{\lambda}\right)$, $\lambda>0$

The formulas nrs 8 and 9 are sometimes called the "*shifting*"-*rules*, nr 10 the "*uniformity*"-*rule.*

All these formulas are easily proved by means of the integral of definition (1). Some of them, viz nrs 1, 2, 5 and 6 have already been deduced. As instructive exercises upon the general methods of the preceding chapters we are now going to verify and discuss several of them in different ways.

At first we compute the integral of formula 7 by residue calculus, after completing, for $t>0$, the line of integration L by an infinite half-circle convex to the left. The poles are $p=-\beta\pm i\omega$, and integrating around these we obtain

$$2\cdot Re\left\{\frac{i\omega\cdot\sin\varphi+\omega\cdot\cos\varphi}{-\beta+i\omega-(-\beta-i\omega)}\cdot e^{-\beta t+i\omega t}\right\}$$

$$=\frac{1}{\omega}\cdot Re\{(\omega\sin\varphi-i\omega\cos\varphi)\,(\cos\omega t+i\sin\omega t)\cdot e^{-\beta t}\}=e^{-\beta t}\cdot\sin(\omega t+\varphi).$$

This rule can also be deduced from the formulas 5, 6, 8 and 2. First we get:

$$\sin\,(\omega t+\varphi)\cdot H\,(t)=\cos\varphi\cdot\sin\omega t\cdot H\,(t)+\sin\varphi\cdot\cos\omega t\cdot H\,(t)$$

$$=p\cdot\frac{p\cdot\sin\varphi+\omega\cdot\cos\varphi}{p^2+\omega^2}\cdot H\,(t)$$

Then, according to 8, we can make the transformation

$$\sin\,(\omega t+\varphi)\cdot H\,(t)=\Phi\,(p)\cdot H\,(t)=\Phi\,(p)\cdot e^{\beta t}\cdot(e^{-\beta t}\cdot H\,(t))$$

$$=e^{\beta t}\cdot\Phi(p+\beta)\cdot e^{-\beta t}\cdot H\,(t).$$

Hence

$$e^{-\beta t}\cdot\sin\,(\omega t+\varphi)\cdot H\,(t)=\Phi\,(p+\beta)\cdot\frac{p}{p+\beta}\cdot H\,(t).$$

Inserting here the expression for $\Phi\,(p+\beta)$ we get formula 7.

The rule nr 8 can be proved e. g. in the following way:

$$\Phi(p+a)\cdot f\,(t)=\Phi(p+a)\cdot F\,(p)\cdot H\,(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Phi\,(\lambda+a)\cdot F\,(\lambda)\cdot\frac{e^{\lambda t}}{\lambda}\,d\lambda.$$

By the change of variable $\lambda+a=p$ we get

$$\Phi\,(p+a)\cdot f\,(t)=e^{-at}\cdot\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Phi\,(p)\cdot F\,(p-a)\cdot\frac{e^{pt}}{p-a}\cdot dp$$

keeping the limits of integration unaltered. This is permitted as the value of the integral is invariant against a displacement of L to the right or to the left, provided that L is always situated sufficiently far to the right. According to the rules for integrators, $\Phi(p)$ can be put outside the sign of integration, and thus we obtain

$$\Phi(p+a)\cdot f(t)=e^{-at}\cdot\Phi(p)\cdot\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}F(\lambda-a)\cdot\frac{e^{\lambda t}}{\lambda-a}\cdot d\lambda,$$

again denoting by λ the variable of integration. At last we introduce $p=\lambda-a$, getting

$$\Phi(p+a)\cdot f(t)=e^{-at}\cdot\Phi(p)\cdot e^{at}\cdot\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}F(p)\cdot\frac{e^{pt}}{p}\cdot dp,$$

that is

$$\Phi(p+a)\cdot f(t)=e^{-at}\cdot\Phi(p)\cdot e^{at}\cdot f(t),$$

which is the result desired.

The function $f(t)$ being arbitrary, we only need to replace it by $e^{-at}\cdot f(t)$ in order to pass from the formula 8 to the formula 8 a.

It may be of some interest to show how the rules nrs 1 and 3 can be deduced from the integral operators of the original basic integrators.

In the former $F(p)=\frac{1}{p}$, and the corresponding integral operator, which we may denote by I, accordingly will be:

$$I=\left[\int\limits_0^t d\tau\right] \ldots\ldots\ldots\ldots\ldots\ldots \tag{72}$$

Therefore

$$\frac{1}{p^n}\cdot\varphi(t)=I^n\times\varphi(t)=\int\limits_0^t d\tau_{n-1}\int\limits_0^{\tau_{n-1}}d\tau_{n-2}\cdots\int\limits_0^{\tau_2}d\tau_1\int\limits_0^{\tau_1}\varphi(\tau)\cdot d\tau.$$

We can transform this repeated integral of order n into a single quadrature by making use of the integral kernel corresponding to I^n. Obviously, this kernel may be at once obtained from the first rule, but here we prefer to deduce it from the general formula (51). Putting $f'(x)=1$ we get by this

$$f'_2(t-\tau)=\int_{\tau}^{t} d\xi = t-\tau, \cdots \quad f'_n(t-\tau)=\frac{(t-\tau)^{n-1}}{\underline{|n-1}}.$$

Hence, as $f(0)=0$, the integral kernel becomes

$$K_n(x)=f'_n(x)=\frac{x^{n-1}}{\underline{|n-1}},$$

and accordingly

$$I^n=\left[\frac{1}{\underline{|n-1}}\int_0^t (t-\tau)^{n-1}\,d\tau\right] \dots\dots\dots\dots\dots \quad (73)$$

Thus we have got the formula

$$\frac{1}{p^n}\cdot\varphi(t)=I^n\times\varphi(t)=\frac{1}{\underline{|n-1}}\cdot\int_0^t (t-\tau)^{n-1}\cdot\varphi(\tau)\,d\tau.$$

If here $\varphi(t)$ is replaced by $H(t)$, the first rule of the table is obtained.

In order to derive the third formula in a similar manner we will start from the integrational expression

$$\frac{1}{(p-\alpha)^n}\cdot\varphi(t).$$

From rule nr 4 follows that

$$f(t)\equiv\frac{1}{p-\alpha}\cdot H(t)=\frac{1}{\alpha}(e^{\alpha t}-1)\cdot H(t)$$

and consequently, as $f(0)=0$,

$$F^n=\left[\int_0^t K_n(t-\tau)\,d\tau\right]=\left[\int_0^t f'_n(t-\tau)\,d\tau\right].$$

Further is

$$f'_1(t-\tau)=e^{\alpha(t-\tau)}, \cdots\cdots \quad f'_n(t-\tau)=\frac{1}{\underline{|n-1}}\cdot(t-\tau)^{n-1}\cdot e^{\alpha(t-\tau)}.$$

Therefore

$$\frac{1}{(p-\alpha)^n}\cdot\varphi(t)=F^n\times\varphi(t)=\frac{1}{\underline{|n-1}}\cdot\int_0^t(t-\tau)^{n-1}\cdot e^{\alpha(t-\tau)}\cdot\varphi(\tau)\cdot d\tau.$$

If here we put $\varphi(t)=H(t)$ and take $u=t-\tau$ as variable of integration, we get for $t>0$

$$\frac{1}{(p-\alpha)^n}\cdot H(t)=\frac{1}{\underline{|n-1}}\cdot\int_0^t u^{n-1}\cdot e^{\alpha u}\cdot du.$$

Now we only have to add the factor $p=\frac{d}{dt}$ to both the members to recover the rule in question.

Of course the employment of residue calculus will bring us the result in the simplest way. Indeed, from the expansion

$$e^{pt}=e^{\alpha t}\cdot e^{t(p-\alpha)}=e^{\alpha t}\left[1+\frac{t}{\underline{|1}}\cdot(p-\alpha)+\frac{t^2}{\underline{|2}}(p-\alpha)^2+\cdots\right]$$

we find at once $\frac{t^{n-1}}{\underline{|n-1}}\cdot e^{\alpha t}$ to be the residue of the pole α.

* * *

At last we will give some examples of different ways of treating a simply composed integrational expression in order to elucidate further the relationship between integrators and integral operators.

If a sinusoidal electromotive force is suddenly applied to an inductance coil, a current $x(t)$ will arise according to the equation

$$x(t)=\frac{\omega\cdot p^2}{(p+\beta)\,(p^2+\omega^2)}\cdot H(t)\;..........\;(74)$$

To obtain the explicit time function one can e. g. proceed as follows:

A. Complex integration.

For positive values of t we have

$$x(t)=\frac{1}{2\pi i}\int_{\text{ɔ}}\frac{\omega p}{(p+\beta)\,(p^2+\omega^2)}\cdot e^{pt}\cdot dp.$$

The poles are $p=-\beta$ and $p=\pm i\omega$. Thus, by residue calculus,

$$x(t)=\omega\left[\frac{-\beta}{\omega^2+\beta^2}\cdot e^{-\beta t}+2\,Re\left\{\frac{i\omega}{(\beta+i\omega)\cdot 2\,i\omega}\cdot e^{i\omega t}\right\}\right]$$

and accordingly

$$x(t)=\frac{\omega}{\omega^2+\beta^2}\cdot(-\beta\cdot e^{-\beta t}+\beta\cdot\cos\omega t+\omega\sin\omega t)\cdot H(t) \quad \ldots\ldots (75)$$

B. Partial fraction decomposition.

By well-known methods one finds

$$\frac{\omega p^2}{(p+\beta)\,(p^2+\omega^2)}=\frac{\omega}{\omega^2+\beta^2}\cdot\left(-\beta\cdot\frac{p}{p+\beta}+\beta\cdot\frac{p^2}{p^2+\omega^2}+\omega\cdot\frac{\omega p}{p^2+\omega^2}\right).$$

Then $x(t)$ is readily obtained from the rules 2, 5 and 6 in the table.

This manner of proceeding illustrates the use of the so-called *expansion theorem*, which, if applicable, will generally offer the simplest way of interpreting an integrational expression. The expansion theorem shall be completely treated in a following chapter.

Note. As a variant of this method can be mentioned the expansion of the integrator into a power-series on $\frac{1}{p}$. One gets the Taylor expansion of the time function by applying rule nr 1 to each term of this series.

C. Factorial decomposition.

We write

$$x(t)=\frac{p}{p+\beta}\cdot\frac{\omega p}{p^2+\omega^2}\cdot H(t)\equiv F_1(p)\cdot F_2(p)\cdot H(t)$$

According to rules 2 and 6 we have

$$F_1(p)\cdot H(t)=e^{-\beta t}\cdot H(t),\quad F_2(p)\cdot H(t)=\sin\omega t\cdot H(t),\quad \ldots (76)$$

and consequently the corresponding operators are

$$F_1=[1-\beta\cdot e^{-\beta t}\int_0^t e^{\beta\tau}d\tau],\quad F_2=[\omega\cdot\int_0^t\cos\omega(t-\tau)\,d\tau]\quad \ldots (77)$$

Hence, on account of (48) and (49)

$$x(t)=F_1\times F_2\times H(t)=[f_1(0)\cdot f_2(0)+\int^t K_{12}(t-\tau)\,d\tau]\times H(t),$$

and the integral kernel is

$$K_{12}(x)=\frac{\omega}{\omega^2+\beta^2}\cdot(\omega^2\cdot\cos\omega x-\omega\beta\cdot\sin\omega x+\beta^2\cdot e^{-\beta x})\ldots\ldots \quad (78)$$

Therefore we obtain the following expression for $x(t)$:

$$x(t)=\frac{\omega}{\omega^2+\beta^2}\cdot\int_0^t(\omega^2\cdot\cos\omega u-\omega\beta\cdot\sin\omega u+\beta^2\cdot e^{-\beta u})\cdot du,$$

which is easily identified with the former (75).

D. Mixed method.

Employing the same factorial decomposition as before, we can alternatively present $x(t)$ under the forms

$$x(t)=F_1\times F_2(p)\cdot H(t) \text{ and } x(t)=F_2\times F_1(p)\cdot H(t).$$

Conformably to (76) and (77) these symbolic equations have the meaning:

$$x(t)=(\sin\omega t-\beta\cdot e^{-\beta t}\cdot\int_0^t e^{\beta\tau}\cdot\sin\omega\tau\cdot d\tau)\cdot H(t)\ldots\ldots \quad (79)$$

$$x(t)=(\omega\cdot e^{-\beta t}\cdot\int_0^t e^{\beta\tau}\cdot\cos\omega\tau\cdot d\tau)\cdot H(t).\ldots\ldots\ldots \quad (80)$$

Of course, also these two expressions are but different ways of writing (75).

6. Application of integrators and operators to the solution of differential equations.

As already emphazised in Chapter 1, the practical applicability of the notion of integrator to the solution of differential equations requires these to be linear and to have constant coefficients with respect to the variable of time. In fact, in this case the introduction of integrators makes possible the more or less simplifying procedure of algebraizing.

One can regard the integrational calculus as a generalization of a well-known method using complex vectors to find in a simple and lucid way those solutions of differential equations of the kind mentioned which correspond to stationary sinusoidal disturbances. In the alternating-current theory this method of calculus is usually called the "*symbolic method*"; there it has got an immense practical importance, making it possible to apply formally the notions and laws of the direct-current theory to steady-state sinusoidal processes. By the integrational calculus this will be possible also to transient phenomena.

In the symbolic method every time function is put under the form $F \cdot e^{i\omega t}$, where $F = \mathrm{F} \cdot e^{i\gamma}$ denotes the representative vector of the magnitude in question. The exponential factor $e^{i\omega t}$ can always be removed from the equations, the symbol $\frac{d}{dt}$ being replaced by $i\omega$ and the symbol $\int dt$ by $1/i\omega$. (The initial conditions are left out of consideration). In this way one gets to solve an equation or a system of equations no longer containing the variable of time. At last one finds the explicit form of the quantity desired, on multiplying the solution of this system by $e^{i\omega t}$ and taking e. g. the imaginary part of the complex number thus obtained.

In the integrational calculus the simple sinusoidal function has its counterpart in a class of functions which are characterized, as we know, by the properties of being all furnished with the factor $H(t)$, expressing their identical disappearing for negative values of t, and further of possessing a basic integrator defining them uniquely. This latter symbol $F(p)$ can be regarded as corresponding to the representative vector F in the symbolic method, just as the unit-function $H(t)$ can be said to correspond to the exponential factor $e^{i\omega t}$. From a methodical point of view we can say that the above mentioned procedure of obtaining the explicit time function in the symbolic method is quite analogous to the forming of the product $F(p) \cdot H(t)$ and the evaluating of the integral of definition (1) in the integrational method.

We recapitulate the three main properties upon which the applicability of integrators in solving differential and integro-differential equations is based, viz

1. *The law of uniqueness.*

Two functions with identical basic integrators are themselves identical and conversely.

2. *The law of differentiation.*

The basic integrator of the *nth* derivative $f^{(n)}(t)$ of a function $f(t)$ with the basic integrator $F(p)$ is:

$$p^n \cdot F(p) - [f(0) \cdot p^n + f'(0) \cdot p^{n-1} + \cdots + f^{(n-1)}(0) \cdot p].$$

3. *The law of integration.*

The basic integrator of the function $\int_0^t f(\tau)\, d\tau$ is $\frac{1}{p} \cdot F(p)$, $F(p)$ being the basic integrator of $f(t)$.

As a first example of the practical usefulness of integrational methods we choose the common equation of oscillatory systems with one degree of freedom,

$$\frac{d^2x}{dt^2} + 2\beta \cdot \frac{dx}{dt} + \omega^2 \cdot x = f(t), \quad \ldots\ldots\ldots\ldots\ldots \quad (81)$$

and assume the initial conditions

$$x(+0) = u \text{ and } \frac{dx}{dt}(+0) = v.$$

Writing $x(t) = X(p) \cdot H(t)$ we obtain, on account of the law of differentiation,

$$\left.\begin{aligned} \frac{dx}{dt} &= (pX - pu) \cdot H(t), \\ \frac{d^2x}{dt^2} &= (p^2X - p^2u - pv) \cdot H(t) \end{aligned}\right| \quad \ldots\ldots\ldots\ldots\ldots \quad (82)$$

Now, by introducing $f(t) = F(p) \cdot H(t)$ and employing the law of uniqueness, we get the equality

$$(p^2 + 2\beta p + \omega^2) \cdot X(p) = F(p) + (p^2 + 2\beta p)u + pv.$$

Therefore the required solution of (81) will be

$$x(t) = x_1(t) + x_2(t),$$

where

$$\left.\begin{aligned} x_1(t) &= \frac{F(p)}{p^2 + 2\beta p + \omega^2} \cdot H(t) = \frac{1}{p^2 + 2\beta p + \omega^2} \cdot f(t) \\ x_2(t) &= \frac{up^2 + (2\beta u + v)p}{p^2 + 2\beta p + \omega^2} \cdot H(t). \end{aligned}\right| \quad \ldots\ldots \quad (83)$$

If $F(p)$ is previously known and of a simple constitution, as e. g. in the case:

$$f(t)=e_0\cdot\sin\omega_0 t\cdot H(t),\quad F(p)=e_0\cdot\frac{\omega_0 p}{p^2+\omega_0^2},$$

the first of the two forms of $x_1(t)$ is the most convenient for the following treatment.

We may easily interpret the integrational expressions obtained and write down the explicit time functions if at first we determine the auxiliary function

$$a(t)=\frac{p}{p^2+2\beta p+\omega^2}\cdot H(t)=\frac{p}{(p+\beta)^2+\omega_1^2}\cdot H(t) \quad \dots. \quad (84)$$

$$\omega_1^2=\omega^2-\beta^2.$$

According to rule nr 7 in the table

$$a(t)=\frac{1}{\omega_1}\cdot e^{-\beta t}\cdot\sin\omega_1 t\cdot H(t). \quad \dots\dots\dots\dots \quad (85)$$

Introducing for a moment

$$b(t)=\frac{1}{p^2+2\beta p+\omega^2}\cdot H(t)=\frac{1}{p}\cdot a(t)=\int_0^t a(\tau)\,d\tau, \quad \dots. \quad (86)$$

we can write the corresponding integral operator

$$B=\left[b(0)+\int_0^t b'(t-\tau)\cdot d\tau\right]=\left[\frac{1}{\omega_1}\cdot\int_0^t e^{-\beta(t-\tau)}\cdot\sin\omega_1(t-\tau)\cdot d\tau\right]. \quad (87)$$

Hence we get

$$x_1(t)=\frac{1}{\omega_1}\cdot\int_0^t e^{-\beta(t-\tau)}\cdot\sin\omega_1(t-\tau)\cdot f(\tau)\cdot d\tau$$

or

$$x_1(t)=\frac{1}{\omega_1}\cdot\int_0^t e^{-\beta\tau}\cdot\sin\omega_1\tau\cdot f(t-\tau)\cdot d\tau. \quad \dots\dots\dots \quad (88)$$

Further, as for $t>0$ the factor p means the same as the symbol $\frac{d}{dt}$, $x_2(t)$ can be obtained according to the equation

$$x_2(t)=\left(u\cdot\frac{d}{dt}+2\beta\cdot u+v\right)\cdot\frac{1}{\omega_1}\cdot e^{-\beta t}\cdot\sin\omega_1 t,$$

from which

$$x_2(t)=\left(\frac{v+\beta u}{\omega_1}\cdot\sin\omega_1 t+u\cdot\cos\omega_1 t\right)\cdot e^{-\beta t}\cdot H(t)\text{....} \quad (89)$$

In studying transient phenomena in electrical circuits built up by a finite number of lumped resistances, capacities, and inductances one will arrive at systems of integro-differential equations of the type

$$\sum_{\nu=1}^{n}\left(L_{\mu\nu}\cdot\frac{di_\nu}{dt}+R_{\mu\nu}\cdot i_\nu+\frac{1}{C_{\mu\nu}}\cdot\int i_\nu dt\right)=e_\mu(t) \quad\text{......}\quad (90)$$

$$(\mu=1,2\cdots n)$$

Indeed, the network can always be divided into a number of meshes, coupled to one another by mutual conductivity, capacity and inductivity, and to each circuit Kirchhoff's law may be applied. In (90) i_ν denotes the current and e_ν the electromotive force in the νth mesh. $R_{\mu\nu}$, $L_{\mu\nu}$ and $C_{\mu\nu}$ are coefficients of resistance, inductance and capacitance respectively. We will assume the system not to be in equilibrium but without initial charges on the capacities at the moment when the forces attack it.*

Introducing the integrators of the currents and forces, one gets the following linear algebraical system for the unknown basic integrators I_ν:

$$\sum_{\nu=1}^{n}Z_{\mu\nu}\cdot I_\nu(p)=E_\mu(p)+\sum_{\nu=1}^{n}p\cdot L_{\mu\nu}\cdot i_\nu(0), \quad\text{........}\quad (91)$$

$$(\mu=1,2\cdots n)$$

where

$$Z_{\mu\nu}=p\cdot L_{\mu\nu}+R_{\mu\nu}+\frac{1}{p\cdot C_{\mu\nu}}\text{..............} \quad (92)$$

---

* The effect of initial charges must be the same as that of constant electromotive forces.

Denoting by $D(p)$ the determinant $|Z_{\mu\nu}|$ and by $D_{\mu\nu}(p)$ the minor, corresponding to $Z_{\mu\nu}$, we can write

$$\left.\begin{aligned} I_\nu(p) &= I'_\nu(p)+I''_\nu(p), \\ I_\nu'(p) &= \frac{1}{D(p)}\sum_{\mu=1}^{n} E_\mu(p)\cdot D_{\mu\nu}(p), \\ I_\nu''(p) &= \frac{p}{D(p)}\cdot\sum_{\mu=1}^{n}\left(\sum_{k=1}^{n} L_{\mu k}\cdot i_k(0)\right)\cdot D_{\mu\nu}(p). \end{aligned}\right\} \quad \ldots\ldots (93)$$

$$(\nu=1,2\cdots n)$$

Thus the integrators appearing are all rational functions of p if, as usual, the $E_\mu(p)$ are rational, and obviously this must always be the case in problems leading to systems of ordinary linear differential equations with constant coefficients. As we shall devote a following chapter to the treatment of this practically most important class of integrators, we may leave aside for the present a nearer analysis of the results gained.

In applications to partial differential equations one meets integrational expressions of various kinds and, as a rule, of a more complicated nature. In order to illustrate how a problem involving initial as well as boundary conditions can be treated by integrational and operational methods we are now going to make some researches on the physically and technically very important wave equation. In this chapter, however, we will only discuss the one-dimensional case, but we will start from the most general form of this equation, represented by the universal equation of telegraphy.

Thus we take under consideration a homogeneous transmission line, the primary constants of which may be: r = the resistance, a = the leakage, l = the inductance and c = the capacitance, all per unit length. The distance between the branches may be d. We assume the line to be exposed to an external magnetic field with the component $\mathrm{B}(x, t)$ of induction normal to the plane of the line and to an external electric field with the component $\mathrm{E}(x, t)$ in this plane, normal to the conductors.

$v\,(x, t)$ may be the voltage between the branches at a point on the distance x from one end at the time t, and $i\,(x, t)$ the corresponding current. The total length of the line may be s. Then we have the following system of differential equations:

$$\left.\begin{aligned} -\frac{\partial v}{\partial x} &= r\cdot i + l\cdot\frac{\partial i}{\partial t} + d\cdot\frac{\partial \mathrm{B}}{\partial t} \\ -\frac{\partial i}{\partial x} &= a\cdot v + c\cdot\frac{\partial v}{\partial t} - c\cdot d\frac{\partial \mathrm{E}}{\partial t} \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (94)$$

According to the fundamental rules we get from these equations for the basic integrators $V = V(x; p)$, $I = I\,(x; p)$, and $B = B\,(x; p)$, $E = E\,(x; p)$ a system of ordinary differential equations, viz

$$\left.\begin{aligned} -\frac{dV}{dx} &= r\cdot I + l\cdot p\cdot[I - i\,(x, 0)] + d\cdot p\cdot[B - \mathrm{B}\,(x, 0)] \\ -\frac{dI}{dx} &= a\cdot V + c\cdot p\cdot[V - v(x, 0)] - c\cdot d\cdot p[E - \mathrm{E}(x, 0)] \end{aligned}\right\} \quad \cdots \quad (95)$$

Note. To simplify our mode of writing we occasionally let capital letters mean basic integrators and not, as usual, integral operators.

We make use of the abbreviations

$$\left.\begin{aligned} \Phi\,(x; p) &= d\cdot p\cdot B\,(x; p) - p\,[l\cdot i(x, 0) + d\cdot \mathrm{B}\,(x, 0)], \\ \Psi\,(x; p) &= -d\cdot c\cdot p\cdot E\,(x; p) - c\cdot p[v(x, 0) - d\cdot \mathrm{E}(x, 0)] \end{aligned}\right\} \quad \cdots \quad (96)$$

thus getting (95) under the more lucid form

$$\left.\begin{aligned} -\frac{dV}{dx} &= (r + pl)\cdot I + \Phi, \\ -\frac{dI}{dx} &= (a + pc)\cdot V + \Psi. \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (97)$$

Hence, eliminating one of the unknown quantities

$$\left.\begin{aligned} \frac{d^2V}{dx^2} - \delta^2\cdot V &= -F_1(x; p), \\ \frac{d^2I}{dx^2} - \delta^2\cdot I &= -F_2(x; p), \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (98)$$

where

$$\delta^2 = (r + pl)\,(a + pc) \cdots\cdots\cdots\cdots\cdots\cdots \quad (99)$$

$$\left.\begin{aligned} -F_1\,(x; p) &= (r + pl)\cdot\Psi\,(x; p) - \Phi'_x\,(x; p), \\ -F_2\,(x; p) &= (a + pc)\cdot\Phi(x; p) - \Psi'_x\,(x; p). \end{aligned}\right\} \quad \cdots\cdots \quad (100)$$

Thus we have reduced the original problem to that of solving an equation of the form:

$$\frac{d^2V}{dx^2}-\delta^2(p)\cdot V=-F(x;p) \quad \ldots\ldots\ldots\ldots \quad (101)$$

under some given boundary conditions. If $Z'(p)$ and $Z''(p)$ denote the impedances of certain energy-sources at the beginning and the end of the line with the electromotive forces $e'(t)=E'(p)\cdot H(t)$ and $e''(t)=E''(p)\cdot H(t)$ respectively, we get the boundary conditions

$$\left.\begin{aligned} V(0;p)&=E'(p)-Z'(p)\cdot I(0;p),\\ V(s;p)&=-E''(p)+Z''(p)\cdot I(s;p),\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (102)$$

assuming, as usual in the practice, the initial values of the currents and charges of the end systems to be zero.

Now, it is not difficult to prove that it is always possible to form the solution of the general transmission-line problem by superposition of the solutions of two more special problems, viz

A. Homogeneous equations of the type (101) with $F(x;p)=0$ and inhomogeneous boundary conditions:

$$\left.\begin{aligned} V(0;p)&=E'_*(p)+\frac{Z'(p)}{r+pl}\cdot\frac{dV}{dx}(0;p)\\ -V(s;p)&=E''_*(p)+\frac{Z''(p)}{r+pl}\cdot\frac{dV}{dx}(s;p),\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (103)$$

where:

$$\left.\begin{aligned} E'_*(p)&=E'(p)+\frac{Z'(p)}{r+pl}\cdot\Phi(0;p)\\ E''_*(p)&=E''(p)+\frac{Z''(p)}{r+pl}\cdot\Phi(s;p).\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (103\text{ a})$$

B. Inhomogeneous equations of the same type and homogeneous boundary conditions like those above with

$$E'_*=E''_*=0.$$

The solution of the problem first mentioned is easily effected since the two constants of integration can always be determined in such a way as to satisfy (103). One finds

$$V(x;p)=V'(x;p)+V''(x;p), \quad \ldots\ldots\ldots\ldots \quad (104)$$

where

$$\left.\begin{aligned} V'(x;\,p) &= \frac{sh\,\delta\,(s-x)+\dfrac{Z''}{Z}\,ch\,\delta\,(s-x)}{\left(1+\dfrac{Z'Z''}{Z^2}\right)\cdot sh\,\delta\,s+\dfrac{Z'+Z''}{Z}\cdot ch\,\delta\,s}\cdot E'_*(p) \\ -V''(x;\,p) &= \frac{sh\,\delta\,x+\dfrac{Z'}{Z}\cdot ch\,\delta\,x}{\left(1+\dfrac{Z'Z''}{Z^2}\right)\cdot sh\,\delta\,s+\dfrac{Z'+Z''}{Z}\cdot ch\,\delta\,s}\cdot E''_*(p) \end{aligned}\right\} \qquad (104\text{ a})$$

and

$$Z=\sqrt{\frac{r+pl}{a+pc}}\,\ldots\ldots\ldots\ldots\ldots\ldots \qquad (105)$$

In order to get a general solution of the latter problem we conveniently introduce a so-called *Green's function* $K\,(x,\xi;\,p)$, according to the definition:

1. $K\,(x,\xi;\,p)$ is continuous in the interval $(0,s)$, where, only the point $x=\xi$ excepted, it satisfies the homogeneous equation

$$\frac{\partial^2 K}{\partial x^2}-\delta^2\,(p)\cdot K=0.$$

2. $K\,(x,\xi;\,p)$ satisfies the homogeneous boundary conditions (103) with $E'_*=E''_*=0$.

3. The first derivative with respect to x, K'_x, is continuous over the whole interval, the point $x=\xi$ excepted, where we shall have:

$$\left[\frac{\partial K\,(x,\xi;\,p)}{\partial x}\right]_{\xi-0}^{\xi+0}=-1.$$

Now, the Green function being found in one way or another, we obtain the solution of the inhomogeneous equation (101) by a simple quadrature according to the formula

$$V(x;\,p)=\int_0^s K\,(x,\xi;\,p)\cdot F\,(\xi;\,p)\cdot d\xi\,\ldots\ldots \qquad (106)$$

The general expression of the function $K\,(x, \xi; p)$ proves to be:

$$\left.\begin{aligned} K\,(x,\xi;p)&=\frac{1}{\delta}\cdot\frac{\left(sh\,\delta x+\frac{Z'}{Z}\cdot ch\,\delta x\right)\cdot\left[sh\,\delta(s-\xi)+\frac{Z''}{Z}\cdot ch\,\delta\,(s-\xi)\right]}{\left(1+\frac{Z'Z''}{Z^2}\right)\cdot sh\,\delta\,s+\frac{Z'+Z''}{Z}\cdot ch\,\delta\,s}, \\ &\qquad\text{when } x\leq\xi \\ K\,(x,\xi;p)&=\frac{1}{\delta}\cdot\frac{\left[sh\,\delta\,(s-x)+\frac{Z''}{Z}ch\,\delta\,(s-x)\right]\cdot\left(sh\,\delta\,\xi+\frac{Z'}{Z}\cdot ch\,\delta\xi\right)}{\left(1+\frac{Z'Z''}{Z^2}\right)\cdot sh\,\delta\,s+\frac{Z'+Z''}{Z}\cdot ch\,\delta\,s} \\ &\qquad\text{when } x\geq\xi \end{aligned}\right\}\quad(107)$$

Finally, to obtain the explicit time functions we have to interpret the integrational expressions which we obtain when applying $V(x; p)$, given by (104) and (106), to $H\,(t)$. Taking under consideration only the most complicated case (106), we are thus faced with an expression of the form (Cf. Appendix 8):

$$v\,(x, t)=V\,(x, p)\cdot H\,(t)=\int_0^s K\,(x, \xi; p)\cdot F\,(\xi; p)\cdot H\,(t)\cdot d\xi. \quad (108)$$

Here, of course, the expression under the sign of integration can be treated in the bulk according to the general integrational rules, and sometimes this is also the simplest way to arrive at $v\,(x, t)$. However, as $F\,(\xi; p)$ corresponds to a disturbance of external nature, while $K\,(x, \xi; p)$ is an integrator that only depends upon and to a certain degree characterizes the system in question, it is, principally at least, more adequate to treat each factor separately, i. e. to make use of the associated integral operators. Thus we may first determine, e. g. directly by the integral of definition (1), the function

$$k\,(x, \xi; t)=K\,(x, \xi; p)\cdot H\,(t)\,\ldots\ldots\ldots\ldots \quad (109)$$

which brings us the operator

$$K\,(x, \xi)=[k\,(x, \xi; 0)+\int_0^t k'_t\,(x, \xi; t-\tau)\cdot d\tau].\,\ldots. \quad (110)$$

Then, if $F\,(\xi; p)$ is a basic integrator, we can derive its primary function

$$f\,(\xi, t)=F\,(\xi; p)\cdot H\,(t).$$

This being done, we obtain the solution required under the form

$$v(x,t)=\int_0^s K(x,\xi)\times f(\xi,t)\cdot d\xi \,\ldots\ldots\ldots\ldots \quad (111)$$

or, more completely written

$$v(x,t)=\int_0^s k(x,\xi;0)\cdot f(\xi,t)\cdot d\xi+\int_0^s d\xi\int_0^t k'_t(x,\xi;t-\tau)\cdot f(\xi,\tau)\cdot d\tau \quad (111a)$$

In the case where $F(\xi;p)$ is not a basic integrator, the method described can always be applied to those terms or groups of terms in $F(\xi;p)$ which have this property (Cf. formulas (100) and (96)); as to the other terms one must, as we know, replace the factor p by $\frac{d}{dt}$ and postpone to the last the effectuating of occurring differentiations.

Accordingly, the main problem will be the determination of the primary function $k(x,\xi;t)$ of $K(x,\xi;p)$. The universal expression (107) tells us how complicated this task can be. (In the practice the impedances $Z'(p)$ and $Z''(p)$ are usually rational functions of p). Frequently, however, one will have to deal with more special cases, which are more easily accessible to the analysis. Thus, in a great number of problems the expression for $\delta(p)$ is reduced to a simple form, which will cause considerable facilitation. In this respect we may distinguish the following typical cases:

1. *The universal line.*

$$\left.\begin{aligned}\delta&=\frac{1}{u}\cdot(p+\mu_1)^{\frac{1}{2}}\cdot(p+\mu_2)^{\frac{1}{2}}\\ Z&=R\cdot(p+\mu_1)^{\frac{1}{2}}\cdot(p+\mu_2)^{-\frac{1}{2}}\end{aligned}\right\}\ldots\ldots\ldots\ldots \quad (111\text{ a})$$

Here the notations mean:

$$u=\frac{1}{\sqrt{lc}},\ \mu_1=\frac{r}{l},\ \mu_2=\frac{a}{c},\ R=\sqrt{\frac{l}{c}}\ldots\ldots\ldots\ldots \quad (113)$$

2. *Line without energy losses.*

Here we have $r=a=0$, hence $\mu_1=\mu_2=0$ and

$$\delta=\frac{p}{u},\ Z=R \qquad (114)$$

3. *Line without distortion.*

Such a transmission-line is characterized by the property

$$\frac{r}{l}=\frac{a}{c}=\mu_1=\mu_2=\mu.$$

Accordingly,

$$\delta=\frac{1}{u}\cdot(p+\mu),\ Z=R \qquad (115)$$

4. *Cable-type line.*

So we call a line the inductance l and leakage a of which can be neglected. In this case we obtain

$$\delta=\sqrt{rc}\cdot\sqrt{p},\ Z=\sqrt{\frac{r}{c}}\cdot\frac{1}{\sqrt{p}} \qquad (116)$$

In the practice, of course, these last cases will only be approximatively realized.

Also the Green function can be considerably simplified by special boundary conditions. We will present some practically important examples of this.

A. *Line balanced at both ends.*

This means that $Z'=Z''=Z$. Accordingly, by (107):

$$\left.\begin{aligned} K(x,\xi;p)&=\frac{1}{2\delta}\cdot e^{\delta(x-\xi)},\quad x\leq\xi\\ K(x,\xi;p)&=\frac{1}{2\delta}\cdot e^{\delta(\xi-x)},\quad x\geq\xi \end{aligned}\right\} \qquad (117)$$

B. *Line short-circuited at both ends.*

Then $Z'=Z''=0$ and we get from (107)

$$\left.\begin{aligned} K(x,\xi;p) &= \frac{sh\,\delta x\cdot sh\,\delta(s-\xi)}{\delta\cdot sh\,\delta s}, \quad x\leq\xi \\ K(x,\xi;p) &= \frac{sh\,\delta(s-x)\cdot sh\,\delta\xi}{\delta\cdot sh\,\delta s}. \quad x\geq\xi \end{aligned}\right\} \quad \ldots\ldots \quad (118)$$

C. *Line isolated at both ends.*

Then $Z'=Z''=\infty$, and we get the same expression for $K(x,\xi;p)$ as in B, except that in the numerator the sign *sh* should be replaced by *ch*.

At last we consider the special case of formula (104) where $E''_*=0$ and the line is balanced at the end or (which will prove to be principally the same thing) when it is of an infinite length. Then the integrator of the voltage function will be of the form:

$$V(x;p)=\frac{e^{-\delta x}}{1+\frac{Z'}{Z}}\cdot E(p) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (119)$$

Hence:

$$v(x,t)=\frac{e^{-\delta x}}{1+\frac{Z'}{Z}}\cdot e(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (119\,\mathrm{a})$$

Many of the above mentioned special cases are mathematically equivalent to some classical problems in other domains of physics, e. g. the theory of elasticity and the theory of heat. For instance, if we combine case nr 2, which means a line without losses, with the boundary condition litt. B, we get a Green function of the form

$$\left.\begin{aligned} K(x,\xi;p) &= \frac{sh\frac{p}{u}x\cdot sh\frac{p}{u}(s-\xi)}{\frac{p}{u}\cdot sh\frac{p}{u}s}, \quad x\leq\xi \\ K(x,\xi;p) &= \frac{sh\frac{p}{u}(s-x)\cdot sh\frac{p}{u}\xi}{\frac{p}{u}\cdot sh\frac{p}{u}s}, \quad x\geq\xi \end{aligned}\right\} \quad \ldots\ldots \quad (120)$$

which appears precisely in the study of the movement of a homogeneous elastic string with fixed ends or that of the air in an organ pipe, closed at both ends. The cases 2 and C combined give the Green function for a string with transversally freely movable ends or for a pipe with open ends. At last case nr 4, i. e. the cable line, where $\delta = k\sqrt{p}$, has its counterpart in the theory of the flow of heat in a homogeneous slab of infinite length which is isolated to heat against the surrounding medium, its end being kept at a constant temperature.

The expression (120) gives an example of a *meromorphical* basic integrator. In (119) again we meet a comparatively simple *irrational* integrator; this expression presents a *many-valued* function of p.

Though the universal expressions (104) and (107) are very complicated, they can be governed comparatively easily by integrational and operational methods if they be at first decomposed into simpler constituents. Such an analyzing procedure, the aim of which is to lead back the interpretation of a composed integrational expression to already known typical integrators, is characteristic of this branch of calculus. Usually, it will then be necessary to introduce expansions that are not closed, which means another kind of complication, but in many cases the state of things is such as to afford sufficiently good approximation by the use of only a few terms in the series occurring.

Employing the *coefficients of reflection*

$$G' = \frac{Z - Z'}{Z + Z'} \text{ and } G'' = \frac{Z - Z''}{Z + Z''}, \qquad \ldots\ldots\ldots\ldots \quad (121)$$

we can easily transform the general expressions (104 a) for the voltage integrator into the form

$$\left.\begin{aligned} V'(x;p) &= \frac{1 - G''\cdot e^{-2\delta(s-x)}}{1 - G'G''\cdot e^{-2\delta s}} \cdot \frac{e^{-\delta x}}{1 + \dfrac{Z'}{Z}} \cdot E'_{*}(p), \\ -V''(x;p) &= \frac{1 - G'\cdot e^{-2\delta x}}{1 - G'G''\cdot e^{-2\delta s}} \cdot \frac{e^{-\delta(s-x)}}{1 + \dfrac{Z''}{Z}} \cdot E''_{*}(p). \end{aligned}\right\} \qquad (122)$$

In a similar way one can transform the expressions (107) for the Green function, getting

$$\left.\begin{aligned} K(x,\xi;p) &= \frac{e^{-\delta(\xi-x)}}{2\delta}\cdot\frac{(1-G'\cdot e^{-2\delta x})\,(1-G''\cdot e^{-2\delta(s-\xi)})}{1-G'G''\cdot e^{-2\delta s}}, \\ &\qquad \text{when } x \leq \xi \\ K(x,\xi;p) &= \frac{e^{-\delta(x-\xi)}}{2\delta}\cdot\frac{(1-G'\cdot e^{-2\delta\xi})\,(1-G''\cdot e^{-2\delta(s-x)})}{1-G'G''\cdot e^{-2\delta s}}, \\ &\qquad \text{when } x \geq \xi \end{aligned}\right\} \ldots \quad (123)$$

As we see, the same expression $1-G'G''\cdot e^{-2\delta s}$ appears in all the denominators. In the most general case we have

$$\delta = \frac{1}{u}\cdot\sqrt{(p+\mu_1)\,(p+\mu_2)}.$$

Now, in the complex integral (1) the path L between the points $\sigma \pm i\infty$ can be situated anywhere to the right of all singularities. We choose for δ that branch of the square root which all over the path L has a positive real part. Then $|e^{-2\delta s}| < 1$ along L. Further, if the reflection coefficients have suitable properties, e. g. if they are such that $|G'G''| < M$ when $|p|$ is sufficiently large, one can remove L so far to the right that anywhere on this line

$$|G'\cdot G''\cdot e^{-2\delta s}| < 1. \quad \ldots\ldots\ldots\ldots\ldots \quad (124)$$

Then the factor $(1-G'G''e^{-2\delta s})^{-1}$ can be expanded into a power series, and it is permitted to integrate term by term along L. By this means one gets the following series representations of the expressions (122) and (123):

$$v'(x,t) = W'(x;p)\cdot e'_{*}(t), \quad v''(x,t) = W''(x;p)\cdot e''_{*}(t),$$

where

$$W'(x;p) = \frac{1}{1+\dfrac{Z'}{Z}}\cdot$$

$$\left\{\sum_{k=0}^{\infty} G'^{k}\cdot G''^{k}\cdot e^{-\delta(2ks+x)} - \sum_{k=0}^{\infty} G'^{k}\cdot G''^{(k+1)}\cdot e^{-\delta[2(k+1)\cdot s-x]}\right\}$$

$$W''(x,p) = \frac{1}{1+\frac{Z''}{Z}} \cdot \qquad \ldots\ldots \quad (125)$$

$$\left\{ -\sum_{k=0}^{\infty} G'^{k} \cdot G''^{k} \cdot e^{-\delta[(2k+1)s-x]} + \sum_{k=0}^{\infty} G'^{(k+1)} \cdot G''^{k} \cdot e^{-\delta[(2k+1)s+x]} \right\}$$

The series for the Green function becomes, if $x \leq \xi$:

$$K(x, \xi; p) = \frac{1}{2\delta} \cdot$$

$$\left\{ \sum_{k=0}^{\infty} G'^{k} \cdot G''^{k} \cdot e^{-\delta(2ks+\xi-x)} \right.$$

$$-\sum_{k=0}^{\infty} G'^{(k+1)} \cdot G''^{k} \cdot e^{-\delta(2ks+\xi+x)} \qquad \ldots\ldots\ldots \quad (126)$$

$$-\sum_{k=0}^{\infty} G'^{k} \cdot G''^{(k+1)} \cdot e^{-\delta[2(k+1)s-\xi-x]}$$

$$\left. +\sum_{k=1}^{\infty} G'^{k} \cdot G''^{k} \cdot e^{-\delta(2ks-\xi+x)} \right\}.$$

The expansion for the case $x \geq \xi$ is obtained by making x and ξ change places in this expression.

The typical term in both the series (125) and (126) is of the form

$$G(p) \cdot [G'(p)]^{k'} \cdot [G''(p)]^{k''} \cdot e^{-\delta\zeta} \qquad \ldots\ldots\ldots \quad (127)$$

where ζ denotes a *positive* number and $G(p)$ stands for one of the factors in front of the formulas. Accordingly, the solution of the original problem can be presented at once in an explicit form (we include, of course, in this term expressions involving quadratures with respect to t in a finite number), if we already know the interpretation of the integrational expressions

$$G(p) \cdot H(t), \quad G'(p) \cdot H(t), \quad G''(p) \cdot H(t) \text{ and } e^{-\delta(p)\zeta} \cdot H(t).$$

Introducing the corresponding integral operators, we may write* e. g.

$$\left.\begin{aligned}\varphi(t) &= G(p)\cdot[G'(p)]^{k'}\cdot[G''(p)]^{k''}\cdot e^{-\delta\zeta}\cdot f(t)\\ &= G\times[G']^{k'}\times[G'']^{k''}\times e^{-\delta\zeta}\cdot f(t)\end{aligned}\right\} \quad \ldots \; (128)$$

In the special case where the transmission line is distortionless, i. e. $\delta(p) = \frac{1}{u}(p+\mu)$, we thus get, according to rule nr 7 of our table,

$$\varphi(t) = e^{-\frac{\zeta}{u}\cdot\mu}\cdot G\times[G']^{k'}\times[G'']^{k''}\times f\left(t-\frac{\zeta}{u}\right) \quad \ldots\ldots \; (129)$$

Here the symbolic operations ought to be performed in agreement with the general rules of Chapter 3.

In the examples given above we have met some different types of integrational expressions. We will postpone the analogous treatment of the wave equation in three dimensions to a later chapter and now proceed to study more in detail some important classes of integrators in order to get general rules of interpreting the expressions already obtained and other in the practice frequently occurring types.

7. A general theory of rational integrators. The expansion theorem.

As already mentioned, integrators of this important class will always appear in connection with systems of ordinary linear differential equations with constant coefficients. Provided that the forces attacking have rational basic integrators, also the basic integrators of the unknown quantities will of course be rational functions of p, thus having no other singularities than a finite number of poles.

Suppose $F(p)$ to be such a function, regular at the point at infinity. In the neighbourhood of this point we have the expansion

$$F(p) = \sum_{\nu=0}^{\infty}\frac{A_\nu}{p^\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots \; (130)$$

---

* Cf. Appendix 6.

Further, suppose that $F(p)$ has the poles $p_0=0,\ p_1,\cdots\ p_N$ of the orders $k_0,\ k_1\cdots\ k_N$ respectively. About the pole p_n $(n\neq 0)$ an expansion of the following form exists:

$$\{F(p)\}_{p_n}=\sum_{\nu=1}^{k_n}\frac{p}{(p-p_n)^{\nu}}\cdot a_{n\nu}+p\cdot P_n(p-p_n).\ \ldots\ldots \quad (131)$$

$$(n=1,2\cdots N)$$

Here $P_n(p-p_n)$ denotes a power series in $(p-p_n)$.

As to the expansion about the origin we prefer to write it in a somewhat different way, viz

$$\{F(p)\}_0=\sum_{\nu=0}^{k_0}\frac{a_{0\nu}}{p^{\nu}}+p\cdot P_0(p)\ \ldots\ldots\ldots\ldots \quad (132)$$

According to the definition of $f(t)=F(p)\cdot H(t)$ and to Jordan's lemma, $f(t)\equiv 0$ for $t<0$, and for $t>0$ $f(t)$ may be presented by a complex integral along a circle about the origin including all the poles of the integrand:

$$f(t)=\frac{1}{2\pi i}\int_{\bigcirc}F(p)\cdot\frac{e^{pt}}{p}\cdot dp\ldots\ldots\ldots\ldots\ldots \quad (133)$$

$$t>0$$

Obviously, the explicit function $f(t)$ will appear under different forms according to our starting from the expansion about the point at infinity or from those about the poles.

The residue of $\frac{F(p)}{p}\cdot e^{pt}$ at the point ∞ is equal to minus the coefficient of p^0 in the series

$$F(p)\cdot e^{pt}=\sum_{\nu=0}^{\infty}\frac{A_{\nu}}{p^{\nu}}\cdot\sum_{\mu=0}^{\infty}\frac{t^{\mu}}{\underline{|\mu}}\cdot p^{\mu},$$

and therefore we get:

$$f(t)=\left[\sum_{\nu=0}^{\infty}\frac{A_{\nu}}{\underline{|\nu}}\cdot t^{\nu}\right]\cdot H(t)\ \ldots\ldots\ldots\ldots \quad (134)$$

Of course we must obtain the same result by applying term by term rule nr 1 of the table.

Now, on the other hand, we will look at the poles inside the circle of integration. The residue of p_n equals the coefficient of $\frac{1}{p-p_n}$ in the expansion of

$$F(p)\cdot\frac{e^{pt}}{p}=e^{p_n t}\cdot\frac{1}{p}\cdot F(p)\cdot e^{(p-p_n)t}.$$

Introducing the power series of $\frac{F(p)}{p}$ and $e^{(p-p_n)t}$, we get for the product of these functions:

$$\frac{F(p)}{p}\cdot e^{(p-p_n)t}=\sum_{\mu=0}^{\infty}(p-p_n)^{\mu}\cdot\frac{t^{\mu}}{\underline{|\mu}}\cdot\left\{\sum_{\nu=1}^{k_n}\frac{a_{n\nu}}{(p-p_n)^{\nu}}+P_n(p-p_n)\right\}$$

$$=\frac{1}{p-p_n}\cdot\left(a_{n1}+a_{n2}\cdot\frac{t}{\underline{|1}}+\cdots\cdot+a_{nk_n}\cdot\frac{t^{k_n-1}}{\underline{|k_n-1}}\right)+\cdots$$

Hence the residue of the point $p_n\ (n\neq 0)$ becomes

$$e^{p_n t}\cdot\sum_{\nu=1}^{k_n}a_{n\nu}\cdot\frac{t^{\nu-1}}{\underline{|\nu-1}}.$$

For the pole at the origin we get the expansion:

$$\frac{e^{pt}}{p}\cdot\left\{\sum_{\nu=0}^{k_0}\frac{a_{0\nu}}{p^{\nu}}+p\cdot P_0(p)\right\}=$$

$$\frac{1}{p}\cdot\left(a_{00}+a_{01}\cdot\frac{t}{\underline{|1}}+\cdots\cdot\ +a_{0k_0}\cdot\frac{t^{k_0}}{\underline{|k_0}}\right)+\cdots\cdots,$$

and the corresponding residue is

$$\sum_{\nu=0}^{k_0}a_{0\nu}\cdot\frac{t^{\nu}}{\underline{|\nu}}.$$

Therefore, joining the contributions from all the poles we obtain the following form of the function $f(t)$:

$$f(t)=\left[\sum_{\nu=0}^{k_0} a_{0\nu}\cdot\frac{t^{\nu}}{\underline{|\nu}}+\sum_{n=1}^{N}\left\{\sum_{\nu=1}^{k_n} a_{n\nu}\cdot\frac{t^{\nu-1}}{\underline{|\nu-1}}\right\}\cdot e^{p_n t}\right]\cdot H(t). \quad \ldots\ldots \quad (135)$$

In the practice one can usually arrive at this result in a simpler way, at first developing $\frac{F(p)}{p}$ into *partial fractions* according to well-known algebraical methods:

$$\frac{F(p)}{p}=\sum_{\nu=0}^{k_0}\frac{a_{0\nu}}{p^{\nu+1}}+\sum_{n=1}^{N}\sum_{\nu=1}^{k_n}\frac{a_{n\nu}}{(p-p_n)^{\nu}}. \quad \ldots\ldots\ldots\ldots \quad (136)$$

Because $F(p)$ is proposed to be regular at infinity we have no constant term in the first sum. Naturally we may as well directly make the disposition

$$F(p)=\sum_{\nu=0}^{k_0}\frac{a_{0\nu}}{p^{\nu}}+\sum_{n=1}^{N}\sum_{\nu=1}^{k_n}\frac{p}{(p-p_n)^{\nu}}\cdot a_{n\nu}. \quad \ldots\ldots \quad (137)$$

The coefficients a being determined, one has only to form the product $F(p)\cdot H(t)$ and then to apply term by term rules nrs 1 and 3, thus obtaining the expression (135).

This formula (135) in connection with (137) is often called *the general expansion theorem.*

In the most common case $F(p)$ has only simple poles, none of them at the origin. Then the expansion for $F(p)$ becomes

$$F(p)=a_0+\sum_{n=1}^{N}\frac{p}{p-p_n}\cdot a_n, \quad \ldots\ldots\ldots\ldots\ldots \quad (138)$$

and accordingly

$$f(t)=\left(a_0+\sum_{n=1}^{N} a_n\cdot e^{p_n t}\right)\cdot H(t) \quad \ldots\ldots\ldots\ldots \quad (139)$$

Here the coefficients are very simply computed by the formulas:

$$a_0 = F(0), \quad a_n = \frac{1}{p_n} \cdot \lim_{p=p_n} \{(p-p_n) \cdot F(p)\} \quad \ldots. \quad (140)$$

If we write $F(p)$ in the form of a quotient:

$$F(p) = \frac{\Phi(p)}{\Psi(p)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (141)$$

and assume $\Phi(p)$ and $\Psi(p)$ to be polynomials in p, the order of $\Psi(p)$ being at least as high as that of $\Phi(p)$, the formula (139) will get the appearance:

$$\frac{\Phi(p)}{\Psi(p)} \cdot H(t) = \left[\frac{\Phi(0)}{\Psi(0)} + \sum_{n=1}^{N} \frac{\Phi(p_n)}{p_n \cdot \Psi'(p_n)} \cdot e^{p_n t} \right] \cdot H(t) \ldots. \quad (142)$$

Hence, the explicit solution is immediately obtained, as soon as one has found the primary factors of the denominator.

The expression just presented is of a simple form suitable for learning by heart; it is often named the *Heaviside expansion theorem.* We will sometimes call it the *simplified* expansion theorem.

In technical and physical applications $F(p)$ will always be a function with real coefficients, and therefore all complex zeros of $\Psi(p)$ will appear in conjugate couples. If $p_{n', n''} = -\beta \pm i\omega$ means such a pair of roots of $\Psi(p) = 0$, we know that the corresponding terms in (142) must also be conjugate to each other. Hence, with the notation

$$\frac{\Phi(-\beta+i\omega)}{(-\beta+i\omega) \cdot \Psi'(-\beta+i\omega)} = A \cdot e^{i\alpha} \quad \ldots\ldots\ldots \quad (143)$$

their contribution will be

$$\{2A \cdot e^{-\beta t} \cdot \cos(\omega t + \alpha)\} \cdot H(t). \ldots\ldots\ldots\ldots \quad (144)$$

In all oscillatory mechanical or electrical systems not containing internal energy sources β is positive on account of the losses. Thus, any couple of conjugate complex zeros of the denominator will give rise to a damped sinusoidal oscillation, setting in at the moment $t = 0$ with some initial value $(= 2A \cdot \cos\alpha)$.

According to formula (134) the function in question is obtained in the form of a power series in t, which, as a rule, is very adequate for calculating $f(t)$ for small values of t. On the other hand, the representation of $f(t)$ by (135) has the advantage of giving a complete insight into the analytical nature of the function.

Putting $t=0$ in (134) we get from (130)

$$f(0)=A_0=F(\infty).$$

Obviously, this important *theorem*, stating that the initial value of the time function is obtained, by putting $p=\infty$ in its basic integrator, always holds when $F(p)$ is regular at infinity (thus even in cases where $F(p)$ is not a rational function; we will come back to this question in later chapters).*

More special, but often of practical importance, is the following *theorem*, which can be easily deduced from (135) and (137).

If the rational integrator $F(p)$ has all its poles situated in the region to the left of the imaginary axis of the p-plane, the stationary value of the function $f(t)=F(p)\cdot H(t)$ is given by $F(0)$.

8. Extension of the expansion theorem to a class of meromorphical basic integrators.

When $F(p)$ is a *meromorphical* function, no singularities but poles will occur, and the expansion theorem can always be applied formally. Further, the correctness of such a procedure can be investigated by the well-known *partial-fraction theorem* due to *Mittag-Leffler*. In this chapter we shall make some examinations in that respect, confining ourselves for the sake of simplicity to the Heaviside form (139) of the expansion theorem.

We will prescribe the function $F(p)$ to have the following properties:

1. Inside any finite region $F(p)$ has only simple poles $p_1, p_2 \cdots$, which we assume to be numbered in such a way that $|p_1| \leq |p_2| \leq |p_3| \leq \cdots$. The residues of $F(p)$ corresponding to these poles may be a_1, a_2, $a_3 \cdots$.
2. It shall be possible to determine an unlimited sequence of circles C_1, $C_2 \cdots$, their centra being in $(\sigma, 0)$ where σ denotes

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* Cf. Appendix 5.

a fixed and sufficiently large positive number, and the radius R_m tending to infinity with m, so that:

a) none of them passes through a pole,

b) $F(p)$ is uniformly limited on all these circles, which means that $|F(p)| < A$ for all points upon C_m, A being independent of m.

3. The origin is a point of regularity for $F(p)$.
4. All poles of $F(p)$ are situated to the left of a line L parallel to the imaginary axis of the p-plane.

Now, P being any point inside C_m not coinciding with a pole p_ν, according to the theorem of residues:

$$\frac{1}{2\pi i}\int_{C_m}\frac{F(p)}{p-P}\cdot dp = F(P) + \sum_{\nu=1}^{n}\frac{a_\nu}{p_\nu - P}, \qquad (145)$$

the summation being extended over all the poles, in number $n(m)$, enclosed by C_m.

To begin with we transform the left hand side of (145) in the following way:

$$\frac{1}{2\pi i}\int_{C_m}\frac{F(p)}{p-P}\cdot dp = \frac{1}{2\pi i}\int_{C_m}\frac{F(p)}{p}\cdot dp + \frac{P}{2\pi i}\int_{C_m}\frac{F(p)}{p(p-P)}\cdot dp$$

$$= F(0) + \sum_{\nu=1}^{n}\frac{a_\nu}{p_\nu} + \frac{P}{2\pi i}\int_{C_m}\frac{F(p)}{p(p-P)}\cdot dp\cdot$$

Accordingly, (145) can be written

$$F(P) = F(0) + \sum_{\nu=1}^{n}\frac{a_\nu}{p_\nu}\cdot\frac{P}{P-p_\nu} + \frac{P}{2\pi i}\int_{C_m}\frac{F(p)}{p(p-P)}\cdot dp. \qquad (146)$$

The last term in this expression tends to zero when m (and thus also n) tends to infinity; indeed, its order of magnitude will be $^1/R_m$. Hence we have arrived at the expansion of $F(p)$ in partial fractions (Mittag-Leffler's theorem):

$$F(p) = F(0) + \sum_{\nu=1}^{\infty}\frac{a_\nu}{p_\nu}\cdot\frac{p}{p-p_\nu}\cdot \qquad (147)$$

The poles being all simple, the residue a_ν must be

$$a_\nu = \lim_{p=p_\nu} [(p-p_\nu)\cdot F(p)]. \;\ldots\ldots\ldots\ldots\; (148)$$

The theorem (147) is a direct generalization of (138). Putting $\frac{a_\nu}{p_\nu} = a_\nu'$ we get even a formal agreement with the previous formulas (138) and (140).

It is easily verified that the rest-term in (146) tends uniformly to zero when $n \to \infty$, P being situated within a circle about the origin with a fixed but arbitrarily chosen radius ϱ. This follow from the inequality:

$$\left|\frac{P}{2\pi i}\int_{C_m}\frac{F(p)}{p(p-P)}\cdot dp\right| < \frac{\varrho\cdot A\cdot R_m}{(R_m-\sigma)\,(R_m-\varrho-\sigma)}\cdot \;\ldots\; (149$$

Indeed, the right hand side will be $< \varepsilon$ if m (or n) is made sufficiently large, and this independently of the position of P inside the circle C_ϱ.

In order to arrive from (147) at our expansion theorem we will now make use of the fourth condition prescribed for $F(p)$. We denote by Γ a contour composed of a part of the line of *integration L in the definition-formula (1) and a half-circle, convex* to the left, its centre being in $(\sigma, 0)$. Further we assume a number of N poles of $F(p)$ to be situated inside Γ. On account of the uniform convergence in (147) the integration may be performed term by term, and thus we obtain

$$\left.\begin{aligned} \frac{1}{2\pi i}\int_\Gamma F(p)\cdot\frac{e^{pt}}{p}\cdot dp &= \frac{1}{2\pi i}\int_\Gamma\left[F(0)+\sum_{\nu=1}^{\infty}\frac{a_\nu}{p_\nu}\cdot\frac{p}{p-p_\nu}\right]\cdot\frac{e^{pt}}{p}\cdot dp = \\ &= F(0)+\sum_{\nu=1}^{N}\frac{a_\nu}{p_\nu}\cdot e^{p_\nu t}. \end{aligned}\right\}\;(150)$$

Now, making the radius of the half-circle tend to infinity, we also cause N to do the same. Because of the second condition stated for $F(p)$ it must be possible to draw the half-circle in such a way that $|F(p)| < A$ everywhere upon it; hence, by

Jordan's lemma, its contribution to the contour integral must disappear in the limit, provided that $t > 0$, and therefore the integral along Γ has the same value as the integral along L in the definition formula. Accordingly, the expression (150) with $N = \infty$ must represent the function $F(p) \cdot H(t)$ for positive values of t. Then, to be ready with the proof of the equality

$$F(p) \cdot H(t) = \left[F(0) + \sum_{\nu=1}^{\infty} \frac{a_\nu}{p_\nu} \cdot e^{p_\nu t} \right] \cdot H(t), \ldots\ldots \text{(151}$$

we only have to add that in the limit the contribution from a half-circle C_m, convex to the right, must be zero, if $t < 0$, and that the line L, on account of the 4th condition, may be deformed into such an infinite half-circle.

We choose an example from Chapter 6 to elucidate the treatment of an integrator of the type in question. Thus, taking under consideration a homogeneous transmission line without losses, the ends being short-circuited, we will try to find the behaviour of the voltage along the line due to the initial state

$$v(x, 0) = f(x), \quad i(x, 0) = 0. \ldots\ldots\ldots\ldots \text{(152)}$$

We assume no external fields to be present.

According to (96), (98), (100), and (108) the expression of the voltage becomes

$$v(x, t) = u \cdot c \cdot (r + pl) \cdot \int_0^s G(x, \xi; p) \cdot H(t) \cdot f(\xi) \cdot d\xi, \quad .. \text{(153)}$$

where the Green-function (Cf. (120)) has the following form

$$\left.\begin{aligned} G(x, \xi; p) &= -\frac{\operatorname{sh}\frac{p}{u}x \cdot \operatorname{sh}\frac{p}{u}(s-\xi)}{\operatorname{sh}\frac{p}{u}s}; \quad x \leq \xi \\ G(x, \xi; p) &= \frac{\operatorname{sh}\frac{p}{u}(s-x) \cdot \operatorname{sh}\frac{p}{u}\xi}{\operatorname{sh}\frac{p}{u}s}; \quad x \geq \xi \end{aligned}\right\} \ldots\ldots \text{(154)}$$

Thus our first task will be to examine an integrational expression of the type

$$\frac{\Phi(p)}{\Psi(p)}\cdot H(t)=\frac{sh\frac{p}{u}(s-x)\cdot sh\frac{p}{u}\xi}{sh\frac{p}{u}s}\cdot H(t).$$

It is easily confirmed that this meromorphical integrator fulfils the four conditions stated. The poles p_ν are all situated upon the imaginary axis

$$p_\nu=i\cdot\nu\cdot\frac{\pi}{s}u,\quad \nu=\pm1,\ \pm2,\ \ldots.$$

Therefore we get, employing the expansion formula (142),

$$\frac{\Phi(p)}{\Psi(p)}\cdot H(t)=\left[\frac{\Phi(0)}{\Psi(0)}+\sum_{\nu=-\infty}^{+\infty}{}'\frac{\Phi(p_\nu)}{p_\nu\cdot\Psi'(p_\nu)}\cdot e^{p_\nu t}\right]\cdot H(t).\,. \quad (155)$$

The apostrophe on the sign of summation denotes that the subscript-value $\nu=0$ must be omitted. In our case the first term within the brackets disappears. Further, as $sh(ia)=i\cdot\sin a$ and $ch(ia)=\cos a$, we get

$$\frac{\Phi(p_\nu)}{p_\nu\cdot\Psi'(p_\nu)}=\frac{i}{\nu\pi}\cdot\frac{\sin\left[\nu\cdot\frac{\pi}{s}(s-x)\right]\cdot\sin\left(\nu\cdot\frac{\pi}{s}\xi\right)}{\cos\nu\pi}\quad\ldots.\ (156)$$

$$=-\frac{i}{\nu\pi}\cdot\sin\left(\nu\cdot\frac{\pi}{s}x\right)\cdot\sin\left(\nu\cdot\frac{\pi}{s}\xi\right).$$

Hence the expansion theorem gives the expression

$$G(x,\xi;p)\cdot H(t)=\left[\frac{2}{\pi}\sum_{\nu=1}^{\infty}\frac{1}{\nu}\cdot\sin\left(\nu\cdot\frac{\pi}{s}x\right)\cdot\sin\left(\nu\cdot\frac{\pi}{s}\xi\right)\cdot\sin\left(\nu\cdot\frac{\pi}{s}ut\right)\right]\cdot H(t),$$

(157)

which is valid when $x\geq\xi$.

In a quite similar way we get the function $G(x,\xi;p)\cdot H(t)$ when $x\leq\xi$. In this case, according to (154), we must obtain an expression identical with the one just found if x and ξ change places; on the other side this expression (157) is symmetric in x and ξ. Therefore we may conclude that (157) holds in any case.

Now, inserting the results gained into (153) and then interpreting the factor p as the symbol $\frac{d}{dt}$, we get the following explicit form of the voltage

$$v(x,t)=u\cdot c\cdot\left(r+l\cdot\frac{d}{dt}\right)\cdot \tag{158}$$

$$\left\{\frac{2}{\pi}\cdot\int_0^s\sum_{\nu=1}^{\infty}\frac{1}{\nu}\cdot\sin\left(\nu\frac{\pi}{s}x\right)\cdot\sin\left(\nu\frac{\pi}{s}\xi\right)\cdot\sin\left(\nu\frac{\pi}{s}ut\right)\cdot f(\xi)\cdot d\xi\right\}\cdot H(t).$$

The uniform convergence of the trigonometric series in the interval $(0, s)$ permits us to integrate term by term. We denote by a_ν the general Fourier coefficient of $f(x)$, defined by

$$a_\nu=\frac{2}{s}\int_0^s f(\xi)\cdot\sin\left(\nu\frac{\pi}{s}\xi\right)\cdot d\xi \quad \ldots\ldots\ldots\ldots \tag{159}$$

If $f(x)$ be continuous, $f'(x)$ and $f''(x)$ piecewise continuous in $(0, s)$, one can easily verify that $\nu^2\cdot a_\nu$ remains less than a fixed number when ν tends to infinity. Therefore the new series converges uniformly with respect to t, and this will also be the case with the series obtained by differentiating it term by term. Hence it is permitted to put the symbol $\frac{d}{dt}$ under the sign of summation, and thus we get the result:

$$v(x,t)=$$

$$\left\{r\cdot s\cdot\sqrt{\frac{c}{l}}\cdot\sum_{\nu=1}^{\infty}\frac{a_\nu}{\nu\cdot\pi}\cdot\sin\left(\nu\frac{\pi}{s}x\right)\cdot\sin\left(\nu\frac{\pi}{s}ut\right)\right. \quad \ldots\ldots\ldots \tag{160}$$

$$\left.+\sum_{\nu=1}^{\infty}a_\nu\cdot\sin\left(\nu\frac{\pi}{s}x\right)\cdot\cos\left(\nu\frac{\pi}{s}ut\right)\right\}\cdot H(t)$$

Here $u^2=1/lc$, and all the coefficients have the dimension of a voltage.

In passing we note that, the sense of the coefficients being another, the first sum in this expression represents the state of vibration of a freely movable elastic string with fixed ends starting from equilibrium position, the initial velocity being $f(x)$. The last sum in (160), on the other side, gives the movement when $f(x)$ denotes the initial position, the initial velocity being zero at all points of the string. Further we observe that, by a simple transformation of the trigonometric products, one gets $v(x, t)$ presented as a superposition of an infinite number of sinusoidal waves travelling in both directions of the x-axis with the velocity u.

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9. Irrational integrators in the form of powers of p. The notion of fractional integration. Treatment of certain integrators being rational functions of powers of p. Asymptotic expansions.

In the second chapter we deduced the very important rule of calculus (18)

$$\frac{1}{p^n} \cdot H(t) = \frac{t^n}{\underline{|n}} \cdot H(t),$$

being valid when n is a positive integer or zero. Now we are going to try to generalize this rule, investigating the sense of an integrator of the type $\frac{1}{p^s}$, s being any real number. For this purpose we will start from the integral of definition (1), giving

$$\frac{1}{p^s} \cdot H(t) = \frac{1}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{pt}}{p^{s+1}} \cdot dp. \quad \ldots\ldots\ldots\ldots\ldots \quad (161)$$

Because of the infinite extension of the line of integration we must set up the convergence condition

$$s > -1. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\alpha)$$

If s is not an integer, p^{-s} will be a many-valued function of p, the origin being a branch-point. We may, however, obtain a uniform branch of this function by making a cut from the origin to infinity, conveniently along the negative real axis of the p-plane (Cf. fig. 4).

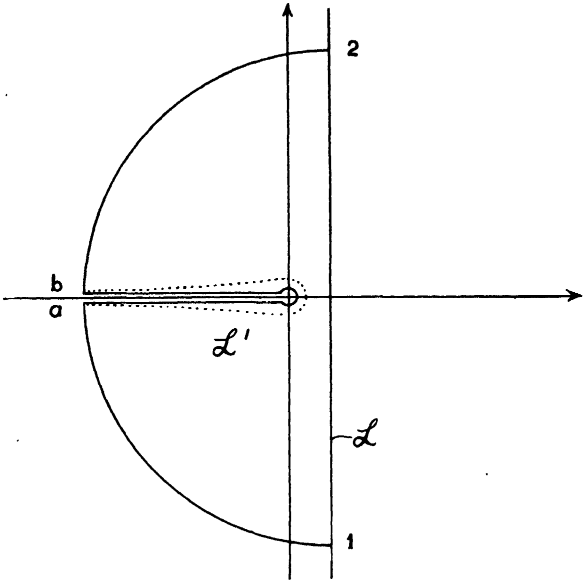

Fig. 4.

The condition (α) being fulfilled, according to Jordan's lemma we get the value zero, if $t<0$, and if we take as path of integration an infinite half-circle convex to the right. On the other hand, if $t > 0$, we get the same result by integrating along an infinite half-circle convex to the left.

Now, let L' be an arbitrary loop starting and ending at the points $-\infty \pm 0 \cdot i$ and enclosing the singularity at the origin. According to Cauchy's theorem we may continuously deform the line L into the new path of integration $(1 \rightarrow a) + L' + (b \rightarrow 2)$ without influencing the value of the integral. The contributions from the quadrants $(1-a)$ and $(b-2)$, as already mentioned, will disappear when $t > 0$. Hence, for positive values of t, we can use the loop L' instead of the line L on evaluating the integral.

In order to lead back the integral along L' to a wellknown function we will now deform the loop until it follows the borders of the cut closely, except at the origin, where it shall have the form of a small circle. If, on this circle, we put $p = \varrho\cdot e^{i\theta}$ and $|e^{pt}| \backsim 1$, we see that its contribution to the value of the integral will be of the order of magnitude ϱ^{-s}. Hence, to get a convergent integral when $\varrho \to 0$, we must prescribe s to be negative.

Accordingly, under the restrictions:

$$-1 < s < 0 \dots\dots\dots\dots\dots\dots \quad (\beta)$$

we may abstract from the small half-circle, thus getting the interpretation

$$\frac{1}{p^s}\cdot H(t) = \frac{1}{2\pi i}\int_{-\infty}^{0} p^{-s-1}\cdot e^{pt}\cdot dp + \frac{1}{2\pi i}\int_{0}^{-\infty} p^{-s-1}\cdot e^{pt}\cdot dp = f(t),$$

below the real axis above the real axis

valid for positive values of t.

To examine this function of t and s we will occasionally put

$$r = -s, \quad 0 < r < +1 \dots\dots\dots\dots\dots \quad (\gamma)$$

Accordingly,

$$f(t) = \frac{1}{2\pi i}\int_{-\infty}^{0} p^{r-1}\cdot e^{pt}\cdot dp + \frac{1}{2\pi i}\int_{0}^{-\infty} p^{r-1}\cdot e^{pt}\cdot dp.$$

below the real axis above the real axis

Now, in the cut p-plane we shall use that branch of the function

$$p^{\alpha} = e^{\alpha\cdot\log p}$$

for which the logarithm has its principal value, i. e.

$$\pi \geq Im\,\{\log\, p\} > -\pi.$$

Then, upon the lower border we get, putting $|p| = x$, $\log p = \log x - i\pi$, and upon the upper border $\log p = \log x + i\pi$. The corresponding expressions for p^{r-1} are

$$x^{r-1}\cdot e^{-(r-1)\cdot\pi i} \text{ and } x^{r-1}\cdot e^{+(r-1)\cdot\pi i}.$$

Therefore, as $p = x\cdot e^{\pm i\pi} = -x$, we obtain

$$f(t) = -\frac{1}{2\pi i}\int_{\infty}^{0} x^{r-1}\cdot e^{-(r-1)\cdot\pi i}\cdot e^{-xt}\cdot dx - \frac{1}{2\pi i}\int_{0}^{\infty} x^{r-1}\cdot e^{(r-1)\cdot\pi i}\cdot e^{-xt}\cdot dx$$

or, combining the terms,

$$f(t)=\frac{1}{2\pi i}\cdot\int\limits_0^\infty x^{r-1}\cdot\left(e^{ir\pi}-e^{-ir\pi}\right)\cdot e^{-xt}\cdot dx=\frac{\sin r\pi}{\pi}\cdot\int\limits_0^\infty x^{r-1}\cdot e^{-xt}\cdot dx \quad (162)$$

This integral can readily be expressed by the so-called Γ-function of Legendre or the Π-function of Gauss. Indeed, putting $xt=u$ we get

$$f(t)=\frac{1}{t^r}\cdot\frac{\sin r\pi}{\pi}\cdot\int\limits_0^\infty u^{r-1}\cdot e^{-u}\cdot du \;\ldots\ldots\ldots\ldots \quad (163)$$

Now, if $Re\{z\}>0$, we have by definition

$$\Pi(z-1)=\Gamma(z)=\int\limits_0^\infty u^{z-1}\cdot e^{-u}\cdot du \;\ldots\ldots\ldots\ldots \quad (164)$$

Thus, instead of (162) we may write

$$f(t)=\frac{1}{t^r}\cdot\frac{\sin r\pi}{\pi}\cdot\Pi(r-1) \;\ldots\ldots\ldots\ldots\ldots \quad (165)$$

By integration by parts in the definition formula above we find, provided that $Re\{z\}>0$, the recurrence rule

$$\Pi(z)=z\cdot\Pi(z-1) \;\ldots\ldots\ldots\ldots\ldots\ldots \quad (166)$$

Hence, n being a positive integer,

$$\Pi(z+n)=(z+n)\cdot(z+n-1)\cdots(z+1)\cdot\Pi(z) \;\ldots\ldots \quad (167)$$

Making $z=0$, as $\Pi(0)=\Gamma(1)=1$, we recover the well-known identity

$$\Pi(n)=n(n-1)\cdots 2\cdot 1=\underline{|n} \;\ldots\ldots\ldots\ldots \quad (168)$$

From (166) or (167) we can obtain the analytic continuation of that function $\Pi(z)$ which by (164) has been defined in the region $Re\{z\}>-1$. It will appear that $\Pi(z)$ is meromorphic in the whole z-plane and possesses the simple poles $z=-1, -2, -3\cdots$. Anywhere, save at these poles, the $\Pi(z)$-function thus completed satisfies, besides (166), also the important relation

$$\Pi(z)\cdot\Pi(-z)=\frac{\pi z}{\sin\pi z} \;\ldots\ldots\ldots\ldots\ldots \quad (169)$$

as proved in textbooks on Γ-functions.

Now, in the expression (165) just found for $f(t)$ we will replace the factor $\frac{\sin r\pi}{\pi}\cdot\Pi(r-1)$ by $\frac{1}{\Pi(-r)}$, thus obtaining

$$f(t)=\frac{1}{t^r}\cdot\frac{1}{\Pi(-r)}.$$

At last, returning to the notation $s=-r$, we arrive at the integrational formula desired, viz

$$\frac{1}{p^s}\cdot H(t)=\frac{t^s}{\Pi(s)}\cdot H(t). \qquad\qquad (170)$$

The condition of convergence first stated, i. e. $s>-1$, must still be sustained. The second condition, $s<0$, on the contrary, may now be annihilated, because the critical integral around the small circle has disappeared from the expression. We understand that this condition must be immaterial already from the fact that it will not come in, if we integrate along a loop L' avoiding the origin; its introducing occasionally, however, made it easy to identify a constituent of the unknown function with the Π-function.

Within the interval $-1<s<0$ we have by (170) got defined a function of t not belonging to the general class of functions we have hitherto dealt with, as it tends to infinity when $t\to+0$, but obviously the expression $\frac{1}{p^s}$ must be a basic integrator for this range of s-values too.

Because, if $s>-1$, the function $\frac{F(p)}{p}=\frac{1}{p^{s+1}}$ tends to zero uniformly with regard to arg p when $p\to\infty$, we may for $t\geq t_0>0$ employ the modified integral

$$\frac{1}{2\pi i}\int\limits_{L_1 L_2}\frac{1}{p^{s+1}}\cdot e^{pt}\cdot dp,$$

and here it is permitted to differentiate under the integral sign any number of times with respect to t. Hence, in the modified integral we have got a means of interpreting, for positive values of t, integrational expressions of the type $\frac{1}{p^{s-n}}\cdot H(t)$ $(n=1, 2, \cdots)$. But, s being any real number >-1, we have obviously got a definition of $\frac{1}{p^s}\cdot H(t)$, valid for all real values of s. It is given by (170) if $t>0$; if $t<0$, we postulate the corresponding time-function to vanish identically. It should be noticed that (170) changes into the more special formula (18), if n is a positive integer, and that the right hand side becomes zero for $s=-1, -2\cdots$, as the Π-function gets infinite for these values of the argument.

The integral operator corresponding to $F(p)=p^{-s}$ is

$$F=\left[\frac{1}{\Pi(s-1)}\cdot\int_0^t (t-\tau)^{s-1}\cdot d\tau\right], \quad \ldots\ldots\ldots \quad (171)$$

provided that $s>0$. Now, in Chapter 5 we have shown that an integer power of the symbol of integration

$$I=[\int_0^t dt]$$

should be formed according to the rule

$$I^n=\left[\frac{1}{\underline{n-1}}\cdot\int_0^t (t-\tau)^{n-1}\,d\tau\right],$$

and it is natural to regard the previous formula (171) as a generalization of this result. Hence we are brought to the *definition:*

$$I^s=\left[\frac{1}{\Pi(s-1)}\cdot\int_0^t (t-\tau)^{s-1}\cdot d\tau\right]. \quad \ldots\ldots\ldots \quad (172)$$

Such a power of the symbol I is sometimes called a *fractional integration.* Of course we need to prove that our definition is consistent with the addition theorem for common powers, viz

$$I^r\times I^s=I^{r+s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (173)$$

The proof is readily effected in the following way.

We form, starting from any function $\varphi(t)$, the expression

$$(I^r \times I^s) \times \varphi(t) =$$

$$\left[\frac{1}{\Pi(r-1)} \cdot \int_0^t (t-\xi)^{r-1} \cdot d\xi\right] \times \left[\frac{1}{\Pi(s-1)} \cdot \int_0^t (t-\tau)^{s-1} \cdot d\tau\right] \times \varphi(t) \tag{174}$$

$$= \frac{1}{\Pi(r-1) \cdot \Pi(s-1)} \cdot \int_0^t \varphi(\tau) \cdot d\tau \int_\tau^t (t-\xi)^{r-1} \cdot (\xi-\tau)^{s-1} \cdot d\xi,$$

where the double-integral has been transformed according to Lejeune-Dirichlet's method. Now, the integral

$$J = \int_\tau^t (t-\xi)^{r-1} \cdot (\xi-\tau)^{s-1} \cdot d\xi \quad \ldots\ldots\ldots\ldots \tag{175}$$

is easily treated. Indeed, from simple substitutions results

$$J = (t-\tau)^{r+s-1} \cdot \int_0^1 (1-u)^{r-1} \cdot u^{s-1} \cdot du = (t-\tau)^{r+s-1} \cdot \frac{\Pi(r-1) \cdot \Pi(s-1)}{\Pi(r+s-1)}, \tag{176}$$

on account of a fundamental theorem as to the Π-function. Inserting (176) into (174), we find the following form for the symbolic product:

$$I^r \times I^s = \left[\frac{1}{\Pi(r+s-1)} \cdot \int_0^t (t-\tau)^{r+s-1} d\tau\right] \quad \ldots\ldots \tag{177}$$

But here, by definition, the right hand side is just I^{r+s}.

By the same substitution as above, or directly, we get

$$I^s \times H(t) = \left[\frac{1}{\Pi(s-1)} \int_0^t (t-\iota)^{s-1} \cdot d\tau\right] \times H(t) = \frac{t^s}{\Pi(s)} \cdot H(t), \tag{178}$$

which formula must be identical with (170).

In the case where $s \leq 0$ the symbol I^s will lead to divergent integrals. It is, however, possible to retain formally the definition (172) even in this case, provided that $\varphi(t)$ is of the form

$$\varphi(t) = \sum a_\nu \cdot t^{r_\nu}.$$

In order to better elucidate this fact we will somewhat examine the product $I^s \times \varphi(t)$, assuming $\varphi(t)$ developable into a Taylor-series

$$\varphi(t) = \sum_{\nu=0}^{\infty} \varphi^{(\nu)}(0) \cdot \frac{t^\nu}{\underline{|\nu}} \quad \ldots\ldots\ldots\ldots\ldots \quad (179)$$

Applying (172) to each term and making also use of the equality

$$\int_0^t (t-\tau)^{s-1} \cdot \tau^\nu \cdot d\tau = \frac{\Pi(s-1) \cdot \Pi(\nu)}{\Pi(s+\nu)} \cdot t^{s+\nu}, \quad \ldots\ldots \quad (180)$$

as the right hand side has a meaning for all s and ν, we may write for $t > 0$:

$$I^s \times \varphi(t) = \sum_{\nu=0}^{\infty} \varphi^{(\nu)}(0) \cdot \frac{t^{s+\nu}}{\Pi(s+\nu)} \cdot \quad \ldots\ldots\ldots \quad (181)$$

If here we put $s = 0$, the left hand side will be $I^0 \times \varphi(t) = \varphi(t)$, while the right hand side changes into the given series (179). If we assume $s = n =$ a positive integer, (181) represents the result of an n-fold integration of $\varphi(t)$ between the limits 0 and t. Further, putting $s = -n$, as $\Pi(-1) = \Pi(-2) = \cdots = \infty$, we have

$$I^{-n} \times \varphi(t) = \sum_{\nu=n}^{\infty} \varphi^{(\nu)}(0) \cdot \frac{t^{\nu-n}}{\underline{|\nu-n}} = \varphi^{(n)}(t).$$

If, at last, we assume $s = -n-x$, x being a positive number < 1, we obtain

$$I^{-n-x} \times \varphi(t) = \sum_{\nu=0}^{\infty} \varphi^{(\nu)}(0) \cdot \frac{t^{\nu-n-x}}{\Pi(\nu-n-x)} \cdot$$

Here $\Pi(\nu-n-x)$ will always be a finite number, and the exponent of t will be negative, as long as $\nu < n+x$. Thus, only provided that all the derivatives of $\varphi(t)$ of orders $< n+x$ disappear at the moment $t=0$, the product $I^{-n-x}\times\varphi(t)$ can represent a function $f(t)$ belonging to class nr 1 at least of the first chapter.

In the foregoing we deduced the formula (170) by performing the complex integration in the definition formula (1). However, this rule may be more simply obtained by an opposite proceeding. In fact, computing according to (2) the function $F(p)$, which corresponds to the time function $\frac{t^s}{\Pi(s)}$, we get

$$F(p)=p\cdot\frac{1}{\Pi(s)}\cdot\int_0^\infty \tau^s\cdot e^{-p\tau}\cdot d\tau. \quad \dots\dots\dots\dots \quad (182)$$

Here we put $u=p\tau$ and assume, to begin with, that p is real and positive. Hence:

$$F(p)=\frac{1}{p^s}\cdot\frac{1}{\Pi(s)}\cdot\int_0^\infty u^s\cdot e^{-u}\cdot du.$$

Now, on account of (164), the last integral is identical with $\Pi(s)$, provided that $s>-1$. On the other side $\frac{1}{p^s}$ is an analytic function of p, and this is also the case with the right hand side of (182) within a certain region. But, as these functions coincide upon the positive real axis, they must do so anywhere in their common domain of definition. Therefore they must also present the same analytic function, and we may accordingly put $F(p)=\frac{1}{p^s}$.

The results obtained above furnish us with a means of interpreting certain classes* of more complicated integrational expressions, e. g. such as are rational functions of powers of p. Such an expression

$$F(p)=\frac{\sum a_\nu\cdot p^{s_\nu}}{\sum b_\nu\cdot p^{r_\nu}}, \quad \dots\dots\dots\dots\dots \quad (183)$$

---

* Cf. Appendix 5.

which we will first propose to be a basic integrator, can be written more conveniently under the form

$$F(p)=\frac{a}{p^s}\cdot\frac{1+\sum\alpha_\nu\cdot p^{-\sigma_\nu}}{1-\sum\beta_\nu\cdot p^{-\varrho_\nu}}, \quad \ldots\ldots\ldots\ldots \quad (184)$$

where $s>-1$ and all σ_ν and $\varrho_\nu>0$.

As before we make a cut in the p-plane along the negative real axis and replace the path of integration L by a loop enclosing the cut and wide enough that

$$\left|\sum\beta_\nu\cdot p^{-\varrho_\nu}\right|<1$$

anywhere upon it. When this is the case, we may develop $(1-\Sigma\beta_\nu\cdot p^{-\varrho_\nu})^{-1}$ into a series and then, after multiplying by the numerator, perform the complex integral around the loop* term by term. Of course, we must arrive at the same result by putting at once

$$F(p)\cdot H(t)=\frac{a}{p^s}\cdot(1+\sum\alpha_\nu\cdot p^{-\sigma_\nu})\cdot\sum_{\mu=0}^{\infty}\left(\sum\beta_\nu\cdot p^{-\varrho_\nu}\right)^{\mu}\cdot H(t) \ldots \quad (185)$$

and applying the general power-rule (170) to each term of this expression. The first term of $f(t)$ becomes

$$a\cdot\frac{t^s}{\Pi(s)}.$$

All the others have an exponent $>s$. If $s=0$, we get $f(0)=a$, and it is easily seen that our previously stated rule (Cf. p. 60)

$$F(\infty)=f(0)$$

subsists also for the class of integrators under consideration (Cf. Appendix 5).

If in (184) $s\leq-1$, $F(p)$ is not a basic integrator. The function $p^s\cdot F(p)$, however, satisfies the condition A of Chapter 2, and therefore we may employ the modified integral to get an interpretation of $F(p)\cdot H(t)$ for positive values of t also in this case.

---

* Obviously the loop will enclose all the poles occurring.

As a practical application we will examine the voltage transient on a coil with the resistance R and the inductance L, being suddenly switched to the end of an infinite cable in series with a battery with the resistance R_0 and the electromotive force E. The impedance of such a line is $Z' = \sqrt{\frac{r}{pc}}$, and we get according to Ohm's law the following expression for the voltage:

$$v(t) = \frac{R+pL}{R_0+R+pL+\sqrt{\frac{r}{pc}}}\cdot \mathrm{E}\cdot H(t) \quad \ldots\ldots\ldots \quad (186)$$

or

$$v(t) = \frac{1+\frac{R}{L}\cdot\frac{1}{p}}{1+\frac{R+R_0}{L}\cdot\frac{1}{p}+\sqrt{\frac{r}{c}}\cdot\frac{1}{L}\cdot\frac{1}{p\sqrt{p}}}\cdot \mathrm{E}\cdot H(t).$$

Hence the initial value is $v(0) = \mathrm{E}$. Taking into account only the first four terms, we obtain:

$$\left.\begin{aligned} v(t) &= \\ &\left[1-\frac{R_0}{L}\cdot\frac{1}{p}-\sqrt{\frac{r}{c}}\cdot\frac{1}{L}\cdot\frac{1}{p\sqrt{p}}+\frac{R_0(R+R_0)}{L^2}\cdot\frac{1}{p^2}+\cdots\right]\cdot \mathrm{E}H(t) \\ =&\left[1-\frac{R_0}{L}\cdot t-\sqrt{\frac{r}{c}}\cdot\frac{1}{L}\cdot\frac{t^{\frac{3}{2}}}{\Pi\left(\frac{3}{2}\right)}+\frac{R_0(R+R_0)}{L^2}\cdot\frac{t^2}{\underline{|2}}+\cdots\right]\cdot \mathrm{E}H(t) \end{aligned}\right\} (187)$$

This expansion is fit for the calculation of $v(t)$ for small values of t. The general expression for $v(t)$ is presented, as we know, by a rather complicated complex integral. The integrational method employed may be looked upon as a shorthand way of getting knowledge of the properties of this integral, when t is a small number.

If the origin is the only branch-point of the integrator $F(p)$, the primary function $f(t)$ contains, besides the contributions from the poles, if such are present, a constituent coming from the integration along the cut, viz

$$I(t) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} F(p)\cdot\frac{e^{pt}}{p}\cdot dp \quad \ldots\ldots\ldots\ldots \quad (188)$$

We will now briefly describe a method,* often applicable, of finding the analytical character of the function $I(t)$ for great values of t.

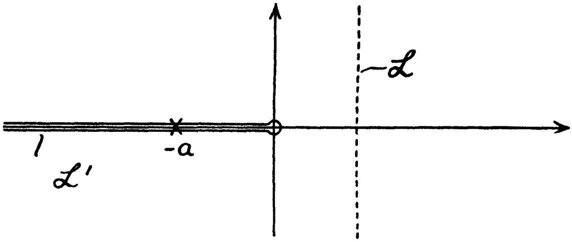

Fig. 5.

If t is a great positive number, the exponential factor will obviously cause only the parts of the loop of integration nearest to the origin to give a considerable contribution to the value of the integral. We divide the loop into three parts with respect to a point $-a$ on the negative real axis (Cf. fig. 5) and denote by $F_u(p)$ the value of $F(p)$ along the lower border of the cut and by $F_0(p)$ its value along the upper border. On the former we have $p=x\cdot e^{-i\pi}=-x$, on the latter $p=x\cdot e^{i\pi}=-x$, $(x\geq 0)$, and thus we may write:

$$I(t)=\frac{1}{2\pi i}\int_a^\infty [F_0(p)-F_u(p)]\cdot\frac{e^{-xt}}{x}\cdot dx+\frac{1}{2\pi i}\int_{-a}^{(0+)} F(p)\cdot\frac{e^{pt}}{p}\cdot dp\,.. \quad (189)$$

Further we assume that it is possible to find two real numbers $K>0$ and k, such that along the loop

$$|F(p)|<K\cdot e^{kx},\text{ when } x\geq a,\; .... \quad (190)$$

a being any positive number.

Then we can easily find an upper limit of the modulus of the first term in (189). In fact,

$$A\equiv\left|\frac{1}{2\pi i}\int_a^\infty [F_0(p)-F_u(p)]\cdot\frac{e^{-xt}}{x}\cdot dx\right|$$
$$<\frac{K}{\pi}\int_a^\infty\frac{e^{-x(t-k)}}{x}\cdot dx<\frac{1}{\pi}\cdot\frac{K}{a}\int_a^\infty e^{-x(t-k)}\cdot dx=\frac{1}{\pi}\cdot\frac{K}{a}\cdot\frac{e^{-a(t-k)}}{t-k},\text{ if } t>k. \quad (191)$$

---

* This method is further developed in Appendix 5.

Considering a, k, and K as given magnitudes, we may find by this formula a number $t_1(\varepsilon)$ such that, ε being an arbitrarily small positive number, the inequality

$$A < \varepsilon$$

holds for all values of t greater than t_1.

Now, to examine the behaviour of $I(t)$ when t is great, we must further assume that in the neighbourhood of the origin $F(p)$ can be written under the form

$$F(p) = \sum_{\nu=1}^{n} a_\nu \cdot p^{s_\nu} + p^s \cdot R_n(p), \quad \ldots\ldots\ldots\ldots \quad (192)$$

where s is a positive number and $s_1 < s_2 < \cdots < s_n$, and the rest-term such that, n being sufficiently large,

$$|R_n(p)| < \frac{\pi \cdot s}{a^s} \cdot \varepsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (192\text{ a})$$

anywhere on the path of integration between the origin and the point $-a$. Then an upper limit of the contribution of the rest-term to $I(t)$ is easily obtained. Indeed,

$$B \equiv \left| \frac{1}{2\pi i} \int_{-a}^{(0+)} p^{s-1} \cdot R_n(p) \cdot e^{pt} \cdot dp \right| < \frac{s}{a^s} \cdot \varepsilon \cdot \int_{0}^{a} x^{s-1} \cdot dx = \varepsilon. \quad (193)$$

Here ε may denote the same positive number as above.

Thus, under the conditions stated, according to (189) and (193) we have, when $t > t_1$:

$$\left| I(t) - \sum_{\nu=1}^{n} a_\nu \cdot \frac{1}{2\pi i} \int_{-a}^{(0+)} p^{s_\nu} \cdot \frac{e^{pt}}{p} \cdot dp \right| < 2\varepsilon. \quad \ldots\ldots\ldots \quad (194)$$

To entertain the convergence when $s_\nu < 0$, we may of course use a loop avoiding the origin.

In the following we shall need an estimation of the magnitude of the difference

$$D = \int_{-\infty}^{(0+)} p^s \cdot e^{pt} \cdot dp - \int_{-a}^{(0+)} p^s \cdot e^{pt} \cdot dp, \quad \ldots\ldots\ldots\ldots \quad (195)$$

s being a real number. Putting, as above, $p=x\cdot e^{i\pi}$ on the upper border and $p=x\cdot e^{-i\pi}$ on the lower one, we get

$$D=-(e^{i\pi s}-e^{-i\pi s})\cdot\int_a^\infty x^s\cdot e^{-xt}\cdot dx,$$

from which follows

$$|D|<2\cdot\int_a^\infty x^s\cdot e^{-xt}\cdot dx. \ldots\ldots\ldots\ldots\ldots \quad (196)$$

Now, if $s\geq 0$, we have for all $\varkappa\geq\frac{s}{e}$ the inequality $x^s\leq e^{\varkappa x}$. Hence, for such values of s,

$$|D|<2\cdot\int_a^\infty e^{x\cdot\left(\frac{s}{e}-t\right)}\cdot dx=2\cdot\frac{e^{-a\left(t-\frac{s}{e}\right)}}{t-\frac{s}{e}},\ \text{if}\ t>\frac{s}{e}\cdot\ldots\ldots \quad (197\ \text{a})$$

In the case where $s<0$, we put $s=-r$, thus obtaining

$$|D|<2\cdot\int_a^\infty\frac{1}{x^r}\cdot e^{-xt}\cdot dx<\frac{2}{a^r}\cdot\int_a^\infty e^{-xt}\cdot dx=\frac{2}{a^r}\cdot\frac{e^{-at}}{t}\ldots \quad (197\ \text{b})$$

We will now consider the difference

$$I(t)-\sum_{\nu=1}^{n}a_\nu\cdot\frac{1}{2\pi i}\int_{-\infty}^{(0+)}p^{s_\nu}\cdot\frac{e^{pt}}{p}\cdot dp$$

$$\equiv I(t)-\sum_{\nu=1}^{n}a_\nu\cdot\frac{1}{2\pi i}\int_{-a}^{(0+)}p^{s_\nu}\cdot\frac{e^{pt}}{p}\cdot dp$$

$$-\sum_{\nu=1}^{n}a_\nu\cdot\frac{1}{2\pi i}\left[\int_{-\infty}^{(0+)}p^{s_\nu-1}\cdot e^{pt}\cdot dp-\int_{-a}^{(0+)}p^{s_\nu-1}\cdot e^{pt}\cdot dp\right].$$

According to (194):

$$\left.\begin{aligned}&\left|I(t)-\sum_{\nu=1}^{n} a_\nu\cdot\frac{1}{2\pi i}\int_{-\infty}^{(0+)} p^{s_\nu}\cdot\frac{e^{pt}}{p}\cdot dp\right|\\ &<2\varepsilon+\sum_{\nu=1}^{n}\frac{|a_\nu|}{2\pi}\cdot\left|\int_{-\infty}^{(0+)} p^{s_\nu-1}\cdot e^{pt}\cdot dp-\int_{-a}^{(0+)} p^{s_\nu-1}\cdot e^{pt}\cdot dp\right|\equiv 2\varepsilon+S.\end{aligned}\right\}\quad(198)$$

Under the sign of summation expressions of the type (195) appear. Let us assume the exponents $s_1, s_2, \cdots s_n$ to be such that

for n_1 of them $\delta_1 > s_\mu - 1 \geq 0$,
for $n_2 = n - n_1$ of them $s_\mu - 1 < -\delta_2 < 0$,

and let α be an upper bound of the magnitudes of the coefficients, i. e. $|a_\nu| < \alpha$, $\nu = 1, 2 \cdots n$.

Then, in accordance with (197 a) and (197 b) we get the following estimation of S in (198):

$$S < \frac{\alpha}{\pi}\cdot\left[n_1\cdot\frac{e^{-a\left(t-\frac{\delta_1}{e}\right)}}{t-\frac{\delta_1}{e}}+n_2\cdot\frac{e^{-at}}{a^{\delta_2}\cdot t}\right], \quad \ldots\ldots\ldots \quad (199)$$

provided that $t > \dfrac{\delta_1}{e}$.

By this inequality a number $t_2(\varepsilon)$ can always be determined so that $S < \varepsilon$ whenever $t > t_2$.

Thus we obtain from (198), taking $t \geq$ the greatest of the numbers t_1 and t_2 just introduced (Cf. (191)),

$$\left|I(t)-\sum_{\nu=1}^{n} a_\nu\cdot\frac{1}{2\pi i}\int_{-\infty}^{(0+)} p^{s_\nu}\cdot\frac{e^{pt}}{p}\cdot dp\right| < 3\varepsilon, \quad \ldots\ldots\ldots \quad (200)$$

ε being an arbitrarily small positive number.

The integrals appearing in this inequality may be written symbolically in the form of integrational expressions, to be interpreted when $s_\nu \geq 1$ by the modified integral for positive values of t. Thus we have arrived at the required "asymptotic" form for the constituent $I(t)$, valid when t is a great number, viz

$$I(t) \backsim \sum_{\nu=1}^{n} a_\nu\cdot p^{s_\nu}\cdot H(t). \quad \ldots\ldots\ldots\ldots\ldots \quad (201)$$

According to (170) this means explicitly

$$I(t) \backsim \sum_{\nu=1}^{n} a_\nu \cdot \frac{t^{-s_\nu}}{\Pi(-s_\nu)}, \quad \ldots\ldots\ldots\ldots \quad (202)$$

and we know that the error committed is less than $3\,\varepsilon$, if $t > t_1, t_2$ and the number of terms is sufficiently large (Cf. (192)).

If the error for any fixed value of n tends to zero as t tends to infinity, we say that $I(t)$ tends *asymptotically* to the right band side of (202). It is easily proved that this may be the case if $s_1 \geq 0$. At first we remark that the contribution to $I(t)$ from the rest term according to (193) is less than

$$\frac{a^s}{\pi \cdot s} \cdot |R_n(p)|$$

and therefore tends to zero with a. Now, let us assume a to approach zero as t increases, putting for example

$$a = \frac{c}{\sqrt{t}}.$$

On introducing this function into the inequality (199) we can infer at once that $S \to 0$ as $t \to \infty$. The same must be the case with the quantity A in formula (191), provided that the condition (190) be fulfilled in the whole interval $x \geq 0$. This, of course, necessitates that $s_1 \geq 0$.

In order to get the function $f(t) = F(p) \cdot H(t)$ we must add to $I(t)$ the contributions from occurring poles. If all the poles are situated in the left half-plane, their contributions must vanish with time, and because of (202) the stationary value of $f(t)$ will be

$f(\infty) = a_1$ if $s_1 = 0$, $f(\infty) = 0$ if $s_1 > 0$, and $f(\infty) = \infty$ if $s_1 < 0$.

But in the neighbourhood of the origin the development (192) subsists, and thus we obtain

$$f(\infty) = F(0),$$

which means an extension of the rule of page 60 to the class of irrational basic integrators in question (Cf. Appendix 5).

As a simple practical application of the method just described we will determine the asymptotic form of the voltage $v(t)$ in the problem outlined above on page 76. For that aim we rewrite the expression (186) in the following more adequate way:

$$v(t)=\sqrt{\frac{pc}{r}}\cdot(R+pL)\cdot\frac{1}{1+(R+R_0)\cdot\sqrt{\frac{c}{r}}\cdot\sqrt{p}+L\sqrt{\frac{c}{r}}\cdot p\sqrt{p}}\cdot \mathrm{E}\cdot H(t) \tag{203}$$

Nearest about the origin we have the expansion

$$F(p)=\mathrm{E}\cdot\sqrt{\frac{c}{r}}\cdot\left\{R\cdot\sqrt{p}-R(R+R_0)\cdot\sqrt{\frac{c}{r}}\cdot p+\right.$$
$$\left.+\left[L+R(R+R_0)^2\cdot\frac{c}{r}\right]\cdot p\sqrt{p}+\cdots\right\},$$

and, as no poles are present, we get for great values of t

$$v(t)\backsim \ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{204}$$
$$\mathrm{E}\cdot\sqrt{\frac{c}{r}}\cdot\left\{R\cdot\frac{t^{-\frac{1}{2}}}{\Pi\left(-\frac{1}{2}\right)}+\left[L+R(R+R_0)^2\cdot\frac{c}{r}\right]\cdot\frac{t^{-\frac{3}{2}}}{\Pi\left(-\frac{3}{2}\right)}+\cdots\right\}$$
$$=\mathrm{E}\cdot\sqrt{\frac{c}{r}}\cdot\frac{1}{\sqrt{\pi}}\cdot\left\{\frac{R}{\sqrt{t}}-\frac{L+R(R+R_0)^2\cdot\frac{c}{r}}{2t\cdot\sqrt{t}}+\cdots\right\}.$$

We let aside, as a matter of pure calculus, the investigation of the domain of validity of this expression at a given degree of approximation, which is of course necessary in exact numerical calculation.

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10. Researches on certain special irrational integrators often occurring in practical applications.

One of the first problems of great practical importance in electrotechnics concerned the transmission properties of a telegraph cable. According to (116) and (119) the behaviour of the voltage along a cable of infinite length is determined by the equation

$$v(x,t)=\frac{e^{-\sqrt{rc}\cdot\sqrt{p}\cdot x}}{1+\sqrt{\frac{c}{r}}\cdot\sqrt{p}\cdot Z'(p)}\cdot e(t)\ \ldots\ldots\ldots\ldots \tag{205}$$

In many cases the impedance $Z'(p)$ of the transmitter can be regarded as constant with respect to p, and often its value is so small that one can equal to 1 the denominator of (205).

Thus we are confronted in this problem with an irrational integrator of the type $e^{-\alpha\sqrt{p}}$, α being a positive number, and our nearest task will be to find the corresponding primary function

$$\varphi(t) = \Phi(p)\cdot H(t) = e^{-\alpha\sqrt{p}}\cdot H(t) \quad \ldots\ldots\ldots\ldots \quad (206)$$

According to the integral of definition (1)

$$\varphi(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{-\alpha\sqrt{p}+pt}.\frac{dp}{p}. \quad \ldots\ldots\ldots\ldots \quad (207)$$

To get a one-valued integrand we make a cut along the negative real axis (Cf. fig. 4). Putting $p=\varrho\cdot e^{i\vartheta}$, where $-\pi \leq \vartheta \leq +\pi$, we have

$$e^{-\alpha\sqrt{p}} = e^{-\alpha\sqrt{\varrho}\cdot\left(\cos\frac{\vartheta}{2}+i\cdot\sin\frac{\vartheta}{2}\right)}.$$

From the equality $|\Phi(p)| = e^{-\alpha\sqrt{\varrho}\cdot\cos\frac{\vartheta}{2}}$ it follows that $\left|\frac{\Phi(p)}{p}\right|$ tends to zero as $|p| \to \infty$, uniformly in the angular region stated. By Jordan's lemma we conclude in a wellknown way that $\varphi(t) \equiv 0$ when $t < 0$, und that for $t > 0$ the line of integration L may be replaced by a loop L' enclosing the cut. Therefore

$$\varphi(t) = \frac{1}{2\pi i}\int_{L'} e^{-\alpha\sqrt{p}+pt}.\frac{dp}{p},\quad t > 0. \quad \ldots\ldots\ldots\ldots \quad (208)$$

Differentiating under the sign of integration, we obtain an integral that converges uniformly along the loop, if $t \geq t_0 > 0$. The integrand being finite at the origin, we may contract the loop until it encloses tightly the cut and then present $\varphi'(t)$ by the integral

$$\varphi'(t) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{-\alpha\sqrt{p}+pt}.dp,\quad t \geq t_0 > 0. \quad \ldots\ldots \quad (209)$$

According to the identity

$$pt - \alpha\sqrt{p} = t\cdot\left[\left(\sqrt{p}-\frac{\alpha}{2t}\right)^2 - \frac{\alpha^2}{4t^2}\right] \quad \ldots\ldots\ldots\ldots \quad (210)$$

this can be rewritten

$$\varphi'(t)=e^{-\frac{\alpha^2}{4t}}\cdot\frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{t\cdot\left(\sqrt{p}-\frac{\alpha}{2t}\right)^2}\cdot dp, \quad \ldots\ldots\ldots\ldots\ldots \quad (211)$$

$$t \geqq t_0 > 0.$$

Now we will employ another variable y, putting

$$\sqrt{tp}-\frac{\alpha}{2\sqrt{t}}=iy, \text{ hence } p=\frac{1}{t}\cdot\left(iy+\frac{\alpha}{2\sqrt{t}}\right)^2 \text{ and}$$

$$dp=\frac{2i}{t}\cdot\left(iy+\frac{\alpha}{2\sqrt{t}}\right)dy \quad \ldots\ldots\ldots\ldots\ldots \quad (212)$$

As to the image of the path of integration in the complex y-plane we note that:

When p moves from $-\infty$ to 0, y moves from $-\infty+\frac{i\alpha}{2\sqrt{t}}$ to $\frac{i\alpha}{2\sqrt{t}}$.

When p moves from 0 to $-\infty$, y moves from $\frac{i\alpha}{2\sqrt{t}}$ to $\infty+\frac{i\alpha}{2\sqrt{t}}$.

Consequently the path of integration in the y-plane must be a line Λ, parallel to the real axis (Cf. fig 6).

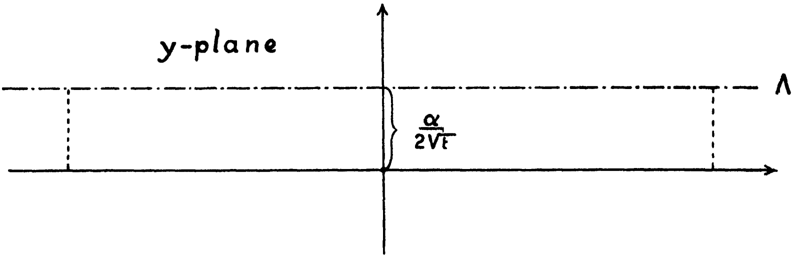

Fig. 6.

The transformed integral looks:

$$\varphi'(t)=e^{-\frac{\alpha^2}{4t}}\cdot\frac{1}{\pi t}\cdot\int_{\Lambda} e^{-y^2}\cdot\left(iy+\frac{\alpha}{2\sqrt{t}}\right)\cdot dy \quad \ldots\ldots \quad (213)$$

$$t \geq t_0 > 0$$

Here the integrand is holomorphical within any region bounded by the real axis, the line Λ, and the segments between them of two lines, parallel to the imaginary axis; as obviously the contributions from the latter boundaries disappear, when their distances from the origin tend to infinity, we conclude that we may take the real axis as path of integration instead of $\mathcal{A}$. Further it is readily seen that the first term under the sign of integration in (213), viz $i\,y \cdot e^{-y^2}$, being an odd function, will give no contribution to the value of the integral. Hence the expression is simplified to

$$\varphi'(t) = \frac{\alpha}{2t\sqrt{t}} \cdot e^{-\frac{\alpha^2}{4t}} \cdot \frac{1}{\pi} \cdot \int\limits_{-\infty}^{+\infty} e^{-y^2} dy = \frac{1}{2\sqrt{\pi}} \cdot \frac{\alpha}{t\sqrt{t}} \cdot e^{-\frac{\alpha^2}{4t}} \quad \ldots\ldots \quad (214)$$

$$t \geq t_0 > 0$$

We are now in a position to introduce the so-called *error-function,* which we will present in the following form, employed by many authors:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \cdot \int\limits_0^x e^{-u^2} \cdot du. \quad \ldots\ldots\ldots\ldots \quad (215)$$

Obviously

$$\frac{d}{dt} \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \cdot e^{-x^2} \cdot \frac{dx}{dt},$$

hence

$$\frac{d}{dt} \operatorname{erf}\left(\frac{\alpha}{2\sqrt{t}}\right) = -\frac{1}{2\sqrt{\pi}} \cdot \frac{\alpha}{t\sqrt{t}} \cdot e^{-\frac{\alpha^2}{4t}}.$$

Here the right hand side equals $\varphi'(t)$ but for the sign. Thus we have

$$\varphi'(t) = -\frac{d}{dt} \operatorname{erf}\left(\frac{\alpha}{2\sqrt{t}}\right). \quad \ldots\ldots\ldots\ldots\ldots \quad (216)$$

Integrating on both sides between the limits 0 and t, we get, as $\operatorname{erf}(\infty) = 1$,

$$\varphi(t) = 1 - \operatorname{erf}\left(\frac{\alpha}{2\sqrt{t}}\right), \; t \geq 0. \ldots\ldots\ldots\ldots \quad (217)$$

Accordingly we have won the new integrational rule:

$$e^{-\alpha\sqrt{p}} \cdot H(t) = \left[1 - \operatorname{erf}\left(\frac{\alpha}{2\sqrt{t}}\right)\right] \cdot H(t) \ldots\ldots\ldots\ldots \quad (218)$$

By (214), as $\varphi(0)=0$, the corresponding integral operator becomes

$$\Phi=\frac{\alpha}{2\sqrt{\pi}}\cdot\left[\int_0^t e^{-\frac{\alpha^2}{4(t-\tau)}}\cdot\frac{d\tau}{(t-\tau)^{3/2}}\right]\cdot \quad \ldots\ldots\ldots \quad (219)$$

For the rest, formula (214) provides us with another important rule:

$$p\cdot e^{-\alpha\sqrt{p}}\cdot H(t)=\frac{1}{2\sqrt{\pi}}\cdot\frac{\alpha}{t\sqrt{t}}\cdot e^{-\frac{\alpha^2}{4t}}\cdot H(t). \quad \ldots\ldots\ldots \quad (220)$$

Thus we see that also the expression $p\cdot e^{-\alpha\sqrt{p}}$ has the character of a basic integrator, and the corresponding primary function vanishes for $t=0$.

Differentiating both members of (218) with respect to α, we obtain one more practical rule of calculus:

$$\sqrt{p}\cdot e^{-\alpha\sqrt{p}}\cdot H(t)=\frac{1}{\sqrt{\pi t}}\cdot e^{-\frac{\alpha^2}{4t}}\cdot H(t). \quad \ldots\ldots\ldots \quad (221)$$

The integrator $e^{-\alpha\sqrt{p}}$ does not admit an expansion into a series of powers of $\frac{1}{p}\cdot$ However, one can easily derive an asymptotic expression for $\varphi(t)$, valid for small values of t. Indeed, by repeated integration by parts in the definition formula (215) one finds, as $\operatorname{erf}(\infty)=1$,

$$\left.\begin{aligned}&1-\operatorname{erf}(x)=\\&\frac{e^{-x^2}}{\sqrt{\pi}}\cdot\left[x^{-1}-\frac{1}{2}\cdot x^{-3}+\frac{1}{2}\cdot\frac{3}{2}\cdot x^{-5}-\ldots\right.\\&\left.+(-1)^{n-1}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{(2n-3)}{2}\cdot x^{-(2n-1)}\right]+R_{2n-1}\end{aligned}\right\} \quad \ldots\ldots \quad (222)$$

the rest-term being:

$$R_{2n-1}=(-1)^n\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{(2n-1)}{2}\cdot\int_{x^2}^{\infty}e^{-y}\cdot y^{-\frac{2n+1}{2}}\cdot dy \qquad (222\text{ a})$$

$$n\geq 2$$

It is readily proved that

$$|R_{2n-1}|<\frac{1}{\sqrt{\pi}}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-3}{2}\cdot e^{-x^2}\cdot x^{-(2n-1)} \quad \ldots\ldots \quad (222\text{ b})$$

From this follows that the rest-term is always numerically less than the last term of the precedent expansion. Thus we have got an asymptotic representation of $1 - \operatorname{erf}(x)$, available if x is a great number.

If specially we insert $x = \frac{\alpha}{2\sqrt{t}}$, we obtain the wanted asymptotic expansion for $\varphi(t)$, viz

$$\varphi(t) \equiv 1 - \operatorname{erf}\left(\frac{\alpha}{2\sqrt{t}}\right) \quad \ldots\ldots\ldots\ldots\ldots \quad (223)$$

$$\backsim \frac{1}{\sqrt{\pi}} \cdot e^{-\frac{\alpha^2}{4t}} \cdot \frac{2\sqrt{t}}{\alpha} \cdot \left[1 - \frac{1}{2} \cdot \frac{4t}{\alpha^2} + \frac{1}{2} \cdot \frac{3}{2} \cdot \left(\frac{4t}{\alpha^2}\right)^2 - \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \left(\frac{4t}{\alpha^2}\right)^3 + \cdots\right],$$

suitable for small values of t or, more strictly speaking, of $\frac{\sqrt{t}}{x}$, as we have $\alpha = x \cdot \sqrt{rc}$, and the error being less than the last term taken into account.

Another asymptotic expression for the function

$$\varphi(t) = e^{-\alpha\sqrt{p}} \cdot H(t),$$

applicable for great values of t, can be deduced by the general method in the foregoing chapter. In fact, upon a loop close to the cut (Cf. fig. 5) $|e^{-\alpha\sqrt{p}}| = 1$, and thus the contribution to the integral from the part of the loop to the left of a point $-a$ will be arbitrarily small if t is great enough. On the other side, this point can be chosen as near to the origin that the function $e^{-\alpha\sqrt{p}}$ can be approximated by a number of terms in the beginning of the corresponding power-series in $\alpha\sqrt{p}$. At last, the path of integration can again be completed by the parts to the left of the point $-a$. Obviously, the result must be the same if we expand formally the integrational expression, putting

$$\varphi(t) \backsim \left[\sum_{\nu=0}^{N} \frac{(-\alpha\sqrt{p})^{\nu}}{\underline{|\nu}}\right] \cdot H(t), \quad \ldots\ldots\ldots\ldots \quad (224)$$

and then, when $t > 0$, interpret each term by the modified integral. Hence we get, according to (170),

$$\varphi(t) \infty \sum_{\nu=0}^{N} \frac{(-\alpha)^{\nu}}{\underline{|\nu}} \cdot \frac{t^{-\frac{\nu}{2}}}{\Pi\left(-\frac{\nu}{2}\right)} \cdot \quad \ldots\ldots\ldots\ldots \quad (225)$$

Here all terms corresponding to even values of ν ($\nu=0$ excepted) disappear, and the first part of the expansion runs

$$\varphi(t) \backsim 1 - \frac{2}{\sqrt{\pi}} \cdot \left[\frac{\alpha}{2\sqrt{t}} - \frac{1}{3} \cdot \left(\frac{\alpha}{2\sqrt{t}}\right)^{3} + \frac{1}{\underline{|2}} \cdot \frac{1}{5} \cdot \left(\frac{\alpha}{2\sqrt{t}}\right)^{5} - \cdots\right] \cdot \quad (225\,a)$$

Referring to the general investigations in Chapter 9, we pass over a more detailed examination of the procedure.

In Chapter 6 has been shown that the treatment of the universal equation of telegraphy leads to integrational expressions of the form $\frac{1}{\delta} \cdot e^{-\zeta \cdot \delta}$, where (Cf. (126)) $\delta = \frac{1}{u} \cdot \sqrt{(p+\mu_1)(p+\mu_2)}$. This expression can readily be led back to an integrator of a more symmetric structure, the primary function of which we are now going to determine. Hence we introduce

$$\chi(t) = \frac{p}{\sqrt{p^2+\omega^2}} \cdot e^{-a\sqrt{p^2+\omega^2}} \cdot H(t), \quad \ldots\ldots\ldots\ldots \quad (226)$$

assuming a positive and employing that branch of the root which tends to p when p tends to infinity.

If we rewrite the expression in the following way:

$$\chi(t) = \frac{p}{\sqrt{p^2+\omega^2}} \cdot e^{-a\left[\sqrt{p^2+\omega^2}-p\right]} \cdot H(t-a), \quad \ldots\ldots\ldots \quad (227)$$

we infer that $\chi(t)=0$ when $t<a$, and further obtain a new integrator

$$\Phi(p) = \frac{p}{\sqrt{p^2+\omega^2}} \cdot e^{-a\left[\sqrt{p^2+\omega^2}-p\right]}, \quad \ldots\ldots\ldots\ldots \quad (228)$$

which satisfies the conditions for Jordan's lemma and obviously is a basic integrator.

We will deduce the explicit time function in two different ways.

At first we develop the exponential function into a series and treat term by term the integrational expression thus appearing.

This is permitted, since $\Phi(p)$ is regular at infinity, as we have chosen the branch of the root. Accordingly we get as a first result:

$$\chi(t)=\sum_{\nu=0}^{\infty}\frac{(-a\omega)^{\nu}}{\underline{|\nu}}\cdot\frac{p}{\sqrt{p^{2}+\omega^{2}}}\cdot\left(\frac{\sqrt{p^{2}+\omega^{2}}-p}{\omega}\right)^{\nu}\cdot H(t-a)\ \ ..\ \ (229)$$

Here we are faced with another important class of basic integrators, which may be defined, more generally, by

$$U(p)=\frac{p}{\sqrt{p^{2}+\omega^{2}}}\cdot\left(\frac{\sqrt{p^{2}+\omega^{2}}-p}{\omega}\right)^{\lambda},\ \ \ldots\ldots\ldots\ \ (230)$$

the exponent λ now being any real number ≥ 0. When λ is not an integer, there exists also a branch-point at infinity. To get a one-valued function we make a cut consisting of two parts, one between the branch-points $\pm\, i\omega$ and another along the negative real axis, as shown by fig. 7. For $\sqrt{p^{2}+\omega^{2}}$ we choose as above that branch which is positive when p is real, and for $(\sqrt{p^{2}+\omega^{2}}-p)^{\lambda}=$ exp $[\lambda\cdot\log\,(\sqrt{p^{2}+\omega^{2}}-p)]$ the branch corresponding to the so-called *principal value* of the logarithm, i. e. that which remains real when p is real.

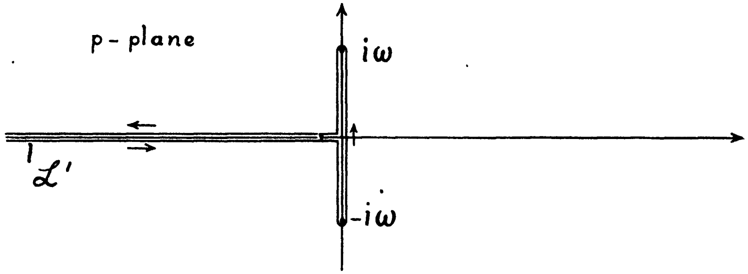

Fig. 7.

When $p\to\infty$ in the cut plane, $U(p)$ tends to zero, if $\lambda>0$, but to 1, if $\lambda=0$. Obviously $U(p)$ is a basic integrator; the primary function $u(t)\equiv 0$, when $t<0$. When $t>0$, employing Jordan's lemma, we may define $u(t)$ by a complex integral extended along a loop L', tightly enclosing the cut (Cf. fig. 7). By the change of variable of integration $p=\omega z$, which geometrically only means a uniform reproduction of the path, we get the expression

$$u(t)=U(p)\cdot H(t)=\frac{1}{2\pi i}\int\limits_{L^*}\frac{(\sqrt{z^2+1}-z)^\lambda}{\sqrt{z^2+1}}\cdot e^{\omega t\cdot z}\cdot dz \quad .. \quad (231)$$

$$t>0$$

Here the right hand side just equals a well-known integral definition (which holds even when λ is complex) of the so-called *Bessel function* of the order λ and the argument ωt. Thus, employing a common notation, we have

$$u(t)=J_\lambda(\omega t),\ \text{when } t>0 \ldots\ldots\ldots\ldots \quad (232)$$

In the special case under consideration (Cf. (229)) only integer values of λ occur. Then there is not a singular point at infinity, and the cut along the negative real axis can be omitted.

Applying our new integrational rule

$$\frac{p}{\sqrt{p^2+\omega^2}}\cdot\left(\frac{\sqrt{p^2+\omega^2}-p}{\omega}\right)^\lambda\cdot H(t)=J_\lambda(\omega t)\cdot H(t);\ \lambda\geq 0 \qquad (233)$$

to the expansion (229), we get the result

$$\chi(t)=\frac{p}{\sqrt{p^2+\omega^2}}\cdot e^{-a\sqrt{p^2+\omega^2}}\cdot H(t)=\sum_{\nu=0}^{\infty}\frac{(-a\omega)^\nu}{\underline{|\nu}}\cdot J_\nu(\omega(t-a))\cdot H(t-a). \qquad (234)$$

We won't discuss more in detail this form of the solution. By introducing into (234) the power-series of $J_\nu(\omega(t-a))$ $(\nu=0,1,2\cdots)$, taken from textbooks of Bessel functions, one shall get a power-series in $\omega(t-a)$ for $\chi(t)$, suitable for numerical evaluation of the function, when $\omega(t-a)$ is small. Of course, one can also obtain the same expansion for $\chi(t)$ by developing $\Phi(p)$ into powers of $\frac{1}{p}$ and then applying the first rule of the collection in Chapter 5.

The second mode of obtaining the function $\chi(t)$ is based upon certain transformations of the definition integral

$$\chi(t)=\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{1}{\sqrt{p^2+\omega^2}}\cdot e^{pt-a\cdot\sqrt{p^2+\omega^2}}\cdot dp. \ \ldots\ldots \quad (235)$$

If $t > a$, we may take as path of integration a loop enclosing the cut between the points $\pm i\omega$ (Cf. fig. 7). We also prefer another variable of integration φ, according to the equation

$$p = i\omega \cdot \sin\varphi. \quad \dots\dots\dots\dots\dots\dots \quad (236)$$

When p moves along the right border of the cut from $-i\,\omega$ to $+i\,\omega$, φ will pass from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ through real values, and again, when p moves along the left border from $i\,\omega$ to $-i\,\omega$, φ will return from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$. Further

$$\pm\sqrt{p^2+\omega^2} = \omega\cos\varphi.$$

On the right side of the cut $\arg(p^2+\omega^2) = 0$, thus $\sqrt{p^2+\omega^2} > 0$; on the left side $\arg(p^2+\omega^2) = 2\pi$ and $\sqrt{p^2+\omega^2} < 0$. At last we have $\frac{dp}{\sqrt{p^2+\omega^2}} = \pm\, i\cdot d\varphi$, the $+$ sign corresponding to the right border, the $-$ sign to the left one. Therefore we get instead of (235),

$$\chi(t) = \frac{1}{2\pi}\cdot\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{i\omega t\cdot\sin\varphi - a\omega\cdot\cos\varphi}\cdot d\varphi - \frac{1}{2\pi}\cdot\int\limits_{\frac{\pi}{2}}^{-\frac{\pi}{2}} e^{i\omega t\cdot\sin\varphi' + a\omega\cdot\cos\varphi'}\cdot d\varphi'$$

or, putting $\varphi' = \pi - \varphi$ in the last integral,

$$\chi(t) = \frac{1}{2\pi}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{i\omega t\cdot\sin\varphi - a\omega\cdot\cos\varphi}\cdot d\varphi + \frac{1}{2\pi}\int\limits_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{i\omega t\cdot\sin\varphi - a\omega\cdot\cos\varphi}\cdot d\varphi.$$

Here the integrals may be joint to one, and as an integration between the limits $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$ because of the periodicity equals an integration between 0 and 2π, we obtain for $t > a$

$$\chi(t) = \frac{1}{2\pi}\cdot\int\limits_{0}^{2\pi} e^{i\omega t\cdot\sin\varphi - a\omega\cdot\cos\varphi}\cdot d\varphi. \quad \dots\dots\dots \quad (237)$$

Into this expression we will introduce an auxiliary imaginary angle θ, defined by the equations

$$t=\sqrt{t^2-a^2}\cdot\sin\theta,\quad i\,a=\sqrt{t^2-a^2}\cdot\cos\theta.$$

Then $i\omega t\cdot\sin\varphi-a\omega\cdot\cos\varphi=i\omega\sqrt{t^2-a^2}\cdot\cos(\varphi-\theta)$ and accordingly

$$\chi(t)=\frac{1}{2\pi}\cdot\int_0^{2\pi}e^{i\omega\cdot\sqrt{t^2-a^2}\cdot\cos(\varphi-\theta)}\cdot d\varphi. \qquad (238)$$

But here we may replace $\varphi-\theta$ by φ, retaining the limits and the path of integration. Indeed, if we put $\varphi-\theta=\varphi'$, the new path between $\varphi'=-\theta$ and $\varphi'=2\pi-\theta$ should be represented by the straight line AB in fig. 8. Because of the periodicity of the cos-function an integration along $0A$ gives the same result as an integration along $2\pi-B$. Further, on account of the regularity of the integrand we must get the value zero on integrating around the parallelogram $0-A-B-2\pi-0$. Therefore, the result must be the same if one integrates along AB or along $0-2\pi$.

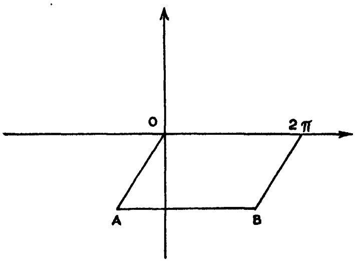

Fig. 8.

Thus we have obtained the required function $\chi(t)$ in the form of a definite integral:

$$\chi(t)=\left[\frac{1}{2\pi}\cdot\int_0^{2\pi}e^{i\omega\cdot\sqrt{t^2-a^2}\cdot\cos\varphi}\cdot d\varphi\right]\cdot H(t-a). \qquad (239)$$

Now the Bessel function of order zero can also be presented under a similar form, viz

$$J_0(x)=\frac{1}{2\pi}\cdot\int_0^{2\pi}e^{ix\cdot\cos\varphi}\cdot d\varphi, \qquad (240)$$

and thus our result can be expressed by the equality

$$\chi(t)=\frac{p}{\sqrt{p^2+\omega^2}}\cdot e^{-a\sqrt{p^2+\omega^2}}\cdot H(t)=J_0\left(\omega\cdot\sqrt{t^2-a^2}\right)\cdot H(t-a). \qquad (241)$$

In the following chapter we shall get use of this formula when studying the general solution of the equation of telegraphy.

As a last example of methods of treating irrational integrators we choose some researches on the expression $\frac{p}{\sqrt{p^2+\omega^2}}$, which often appears e. g. in the theory of electric filters and recurrent networks. As this corresponds to the special case $\lambda=0$ in (233), we know *à priori* that the function

$$u(t)=\frac{p}{\sqrt{p^2+\omega^2}}\cdot H(t), \quad \ldots\ldots\ldots\ldots\ldots \quad (242)$$

when $t \geq 0$, coincides with the Bessel function $J_0(\omega t)$.

This basic integrator, because of its simplicity, is fit for illustrating the above mentioned method of developing $F(p)$ into powers of $\frac{1}{p}$. Thus, writing

$$u(t)=\left[1+\left(\frac{\omega}{p}\right)^2\right]^{-\frac{1}{2}}\cdot H(t),$$

and expanding in accordance with the binomial formula:

$$(1+x^2)^{-\frac{1}{2}}=$$

$$=1-\frac{\underline{|2}}{1}\cdot\left(\frac{x}{2}\right)^2+\frac{\underline{|4}}{(\underline{|2})^2}\cdot\left(\frac{x}{2}\right)^4-\frac{\underline{|6}}{(\underline{|3})^2}\cdot\left(\frac{x}{2}\right)^6+\frac{\underline{|8}}{(\underline{|4})^2}\cdot\left(\frac{x}{2}\right)^8-\ldots, \quad (243)$$

we get on employing term by term the rule $\frac{1}{p^n}\cdot H(t)=\frac{t^n}{\underline{|n}}\cdot H(t)$ the result:

$$u(t)= \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (244)$$

$$\left[1-\frac{1}{(\underline{|1})^2}\cdot\left(\frac{\omega t}{2}\right)^2+\frac{1}{(\underline{|2})^2}\cdot\left(\frac{\omega t}{2}\right)^4-\frac{1}{(\underline{|3})^2}\cdot\left(\frac{\omega t}{2}\right)^6+\frac{1}{(\underline{|4})^2}\cdot\left(\frac{\omega t}{2}\right)^8-\cdots\right]\cdot H(t).$$

Here the brackets just enclose the well-known series for $J_0(\omega t)$.

The correctness of the operations performed is easily verified, if in the definition integral for $t>0$ we integrate e. g. along a circle about the origin, its radius being $>\omega$.

Obviously, the function $u(t)$ to be determined will appear in varying analytical forms according to the path of integration chosen. If we integrate along the loop enclosing the cut between the points $\pm\, i\omega$ (Cf. fig. 7), we get by employing the variable $x=\frac{p}{i\omega}$, when $t>0$,

$$u(t)=\frac{2}{\pi}\int_0^1 \frac{1}{\sqrt{1-x^2}}\cdot\cos[(\omega t)\cdot x]\cdot dx. \quad \ldots\ldots\ldots \quad (245)$$

This definite integral, as well as the one obtained by putting $x=\cos z$, i. e.

$$u(t)=\frac{2}{\pi}\cdot\int_0^{\frac{\pi}{2}} \cos[(\omega t)\cdot\cos z]\cdot dz, \quad \ldots\ldots\ldots \quad (245\text{ a})$$

represents a frequently used definition formula for $J_0(\omega t)$. (Cf. (240)).

Now we are going to examine more in detail another integral form for $u(t)$, which we obtain by using two cuts between $\pm i\omega$ and ∞, parallel to the real axis. (Cf. fig. 9.)

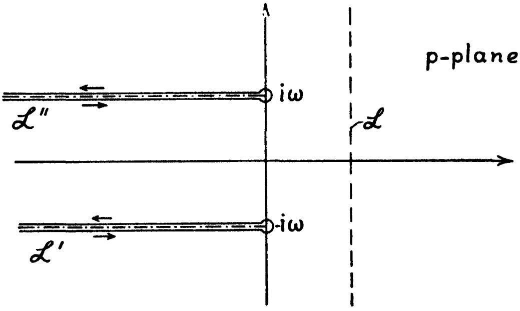

Fig. 9.

For positive values of t we may integrate along the loops L' and L'', tightly enclosing these cuts; next to the branch-points we first assume the loops to have the form of small circles; however, it is easily proved that the contributions from these circles vanish when the radii tend to zero, though the integrand becomes infinite at the points $\pm i\omega$.

Upon the lower loop we put $p=-\beta-i\omega$. Hence

$$p^2+\omega^2=\beta^2+i\cdot 2\beta\omega=\beta\cdot\sqrt{\beta^2+4\omega^2}\cdot e^{i\cdot\operatorname{arc\,tg}\frac{2\omega}{\beta}}.$$

Upon the upper loop we put $p=-\beta+i\omega$. Hence

$$p^2+\omega^2=\beta^2-i\cdot 2\beta\omega=\beta\cdot\sqrt{\beta^2+4\omega^2}\cdot e^{i\cdot\operatorname{arc\,tg}\frac{-2\omega}{\beta}}.$$

Occasionally we will use the abbreviations

$$\frac{1}{\sqrt{\beta}\cdot\sqrt[4]{\beta^2+4\omega^2}}=A(\beta),\ -\frac{1}{2}\cdot\text{arc tg}\,\frac{2\omega}{\beta}=\alpha(\beta).\ \ \ldots.\ (246)$$

Then the expression under the integral sign in the definition formula

$$u(t)=\frac{1}{2\pi i}\int\limits_{L'L''}\frac{e^{pt}}{\sqrt{p^2+\omega^2}}\cdot dp\ \ \ldots\ldots\ldots\ldots\ (247)$$

gets the following form on the different parts of the loop:

$$\begin{array}{lcl}
(-\infty\,|\,-i\omega)\rightarrow(\ 0\ |\,-i\omega) & : & +A(\beta)\cdot e^{-\beta t}\cdot e^{i(-\omega t+\alpha(\beta))}\cdot d\beta \\
(\ 0\ |\,-i\omega)\rightarrow(-\infty\,|\,-i\omega) & : & -A(\beta)\cdot e^{-\beta t}\cdot e^{i(-\omega t+\alpha(\beta))}\cdot d\beta \\
(-\infty\,|\ \ i\omega)\rightarrow(\ 0\ |\ \ i\omega) & : & -A(\beta)\cdot e^{-\beta t}\cdot e^{i(\omega t-\alpha(\beta))}\cdot d\beta \\
(\ 0\ |\ \ i\omega)\rightarrow(-\infty\,|\ \ i\omega) & : & +A(\beta)\cdot e^{-\beta t}\cdot e^{i(\omega t-\alpha(\beta))}\cdot d\beta
\end{array}$$

Combining the different contributions, we therefore obtain:

$$u\ (t)=\frac{2}{\pi}\cdot\int\limits_0^{\infty}A(\beta)\cdot e^{-\beta t}\cdot\sin\left[\omega t-\alpha(\beta)\right]\cdot d\beta\ \ \ldots.\ (248)$$

Hence, our original complex integral (247) now has been transformed into a real integral, and we get for $t>0$ after substituting the expressions (246):

$$u(t)=\frac{2}{\pi}\int\limits_0^{\infty}\frac{1}{\sqrt{\beta}\cdot\sqrt[4]{\beta^2+4\omega^2}}\cdot e^{-\beta t}\cdot\sin\left(\omega t+\frac{1}{2}\ \text{arc tg}\ \frac{2\omega}{\beta}\right)\cdot d\beta.\ \ (249)$$

According to well-known trigonometric formulas this expression can also be written

$$\left.\begin{array}{l}
u(t)= \\
\dfrac{\sqrt{2}}{\pi}\cdot\displaystyle\int\limits_0^{\infty}\frac{\sqrt{\beta+\sqrt{\beta^2+4\omega^2}}}{\sqrt{\beta}\cdot\sqrt{\beta^2+4\omega^2}}\cdot e^{-\beta t}\cdot d\beta\cdot\sin\omega t \\
+\dfrac{\sqrt{2}}{\pi}\cdot\displaystyle\int\limits_0^{\infty}\frac{\sqrt{-\beta+\sqrt{\beta^2+4\omega^2}}}{\sqrt{\beta}\cdot\sqrt{\beta^2+4\omega^2}}\cdot e^{-\beta t}\cdot d\beta\cdot\cos\omega t. \\
t\geq 0
\end{array}\right|\ \ldots.\ (250)$$

From this representation we can easily deduce an asymptotic expression for $u(t)$, available for large values of t. If t is a great positive number, the factor $e^{-\beta t}$ even for small values of β will be so minute that the integrand is practically zero. Next to the origin the factor $\frac{1}{\sqrt{\beta}}$ is dominating. On the other side, the factor $\sqrt{\pm\beta+\sqrt{\beta^2+4\omega^2}} : \sqrt{\beta^2+4\omega^2}$ is continuous at the origin, where it equals $\frac{1}{\sqrt{2\omega}}$. For that reason we might replace the given integrand by the simpler one

$$\frac{1}{\sqrt{2\omega}} \cdot \frac{e^{-\beta t}}{\sqrt{\beta}},$$

retaining the limits 0 and ∞. Then we shall obtain a value of the right hand side that must agree better with the true value the larger t is, thus furnishing us with an asymptotic expression for $u(t)$, viz

$$u(t) \infty \frac{1}{\pi \cdot \sqrt{\omega}} \cdot K(t) \cdot (\sin \omega t + \cos \omega t), \quad \ldots\ldots\ldots \quad (251)$$

where

$$K(t) = \int_0^\infty e^{-\beta t} . \frac{d\beta}{\sqrt{\beta}} . \quad \ldots\ldots\ldots\ldots\ldots \quad (251\text{ a})$$

By the substitution $\beta t = u^2$ the last integral is essentially identified with the so-called *error-integral*, and we find $K(t) = \sqrt{\frac{\pi}{t}}$.

Therefore:

$$u(t) \backsim \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{\omega t}} \cdot \cos\left(\omega t - \frac{\pi}{4}\right) \cdot \quad \ldots\ldots\ldots\ldots \quad (252)$$

Here we recognize on the right hand side a well-known asymptotic form for the function $J_0(\omega t)$ (Cf. e. g. Funktionentafeln, von Jahnke-Emde.)

Of course, by developing the irrational factor under the sign of integration into a power series of $\sqrt{\beta}$ one can arrive at more complete asymptotic expressions for $u(t)$. This shall be done in a somewhat different way in Section 5 of the Appendix.

11. Applications to selected problems of physics and electrotechnics.

In this chapter we are going to employ integrational and operational methods for treating some important problems of theoretical physics and electrotechnics, and we will generally try to work out the solutions in such forms as are already more or less familiar, the comparison of the new methods with older ones thus being facilitated.

A. The differential equation of a freely movable homogeneous elastic string.

If u denotes the elongation, a a constant of the dimension of a velocity, this equation is

$$\frac{\partial^2 u}{\partial t^2}=a^2\cdot\frac{\partial^2 u}{\partial x^2}\cdot\ldots\ldots\ldots\ldots\ldots\ldots \quad (253)$$

We shall now integrate the equation under the initial conditions

$$u=F(x),\ \frac{\partial u}{\partial t}=G(x),\ \text{when } t=0\ldots\ldots\ldots \quad (254)$$

For that aim we introduce a basic integrator depending on x, writing

$$u(x,t)=U(x;p)\cdot H(t).\ \ldots\ldots\ldots\ldots \quad (255)$$

Then, according to the differentiation rule:

$$\left.\begin{aligned}\frac{\partial u}{\partial t}&=[p\cdot U-p\cdot F(x)]\cdot H(t)\\ \frac{\partial^2 u}{\partial t^2}&=[p^2\cdot U-p^2\cdot F(x)-p\cdot G(x)]\cdot H(t)\end{aligned}\right\}\ldots\ldots \quad (256)$$

(Cf. the Note on page 45).

Inserting (256) in (253), we obtain

$$[p^2\cdot U-p^2\cdot F(x)-p\cdot G(x)]\cdot H(t)=a^2\cdot\frac{d^2U}{dx^2}\cdot H(t).$$

Hence, according to the rule of uniqueness we get for the basic integrator the ordinary differential equation:

$$\frac{d^2U}{dx^2}-\frac{p^2}{a^2}U=-\frac{p^2}{a^2}\cdot F(x)-\frac{p}{a^2}\cdot G(x).\ \ldots\ldots\ldots\ldots \quad (257)$$

The corresponding homogeneous equation having the two independent solutions

$$e^{\frac{p}{a}x} \text{ and } e^{-\frac{p}{a}x},$$

we may put, employing the method of varying the constants,

$$U = A(x) \cdot e^{\frac{p}{a}x} + B(x) \cdot e^{-\frac{p}{a}x}. \qquad (258)$$

With the occasional notation

$$k(x) = -\frac{p}{a} \cdot F(x) - \frac{1}{a} \cdot G(x) \qquad (259)$$

we get

$$A'(x) = \frac{1}{2} \cdot e^{-\frac{p}{a}x} \cdot k(x), \quad B'(x) = -\frac{1}{2} \cdot e^{\frac{p}{a}x} \cdot k(x) \qquad (260)$$

Hence the required integral of (257) will be

$$U(x; p) = \frac{1}{2} \cdot e^{\frac{p}{a}x} \cdot \int_{\alpha}^{x} e^{-\frac{p}{a}\xi} \cdot k(\xi) \cdot d\xi - \frac{1}{2} \cdot e^{-\frac{p}{a}x} \cdot \int_{\beta}^{x} e^{\frac{p}{a}\xi} \cdot k(\xi) \cdot d\xi. \qquad (261)$$

Making both the members operate on $H(t)$, we find

$$u(x, t) = -\frac{1}{2} \int_{x}^{\alpha} e^{\frac{p}{a}(x-\xi)} \cdot H(t) \cdot k(\xi) \cdot d\xi - \frac{1}{2} \int_{\beta}^{x} e^{-\frac{p}{a}(x-\xi)} \cdot H(t) \cdot k(\xi) \cdot d\xi$$

But, as we know

$$e^{\frac{p}{a}(x-\xi)} \cdot H(t) = H\left(t + \frac{x-\xi}{a}\right), \quad e^{-\frac{p}{a}(x-\xi)} \cdot H(t) = H\left(t - \frac{x-\xi}{a}\right),$$

and further

$$H\left(t + \frac{x-\xi}{a}\right) \equiv 0 \text{ if } \xi > x + at; \quad H\left(t - \frac{x-\xi}{a}\right) \equiv 0 \text{ if } \xi < x - at.$$

Hence we may write as well

$$u(x, t) = -\frac{1}{2} \cdot \int_{x}^{x+at} k(\xi) \cdot d\xi - \frac{1}{2} \cdot \int_{x-at}^{x} k(\xi) \cdot d\xi. \qquad (262)$$

Inserting $k(\xi)$ from (259) and combining the integrals, we get

$$u(x,t)=\frac{1}{2a}\cdot p\cdot\int\limits_{x-at}^{x+at}F(\xi)\cdot d\xi+\frac{1}{2a}\cdot\int\limits_{x-at}^{x+at}G(\xi)\cdot d\xi. \quad\ldots\ldots (263)$$

Here p is equivalent to the symbol $\frac{d}{dt}$, and we obtain according to the identity

$$\frac{d}{d\alpha}\int\limits_{f(\alpha)}^{\varphi(\alpha)}F(x)\cdot dx=\varphi'(\alpha)\cdot F(\varphi(\alpha))-f'(\alpha)\cdot F(f(\alpha)), \quad\ldots\ldots (264)$$

the final expression for the elongation of the string:

$$u(x,t)=\frac{1}{2}[F(x+at)+F(x-at)]+\frac{1}{2a}\cdot\int\limits_{x-at}^{x+at}G(\xi)\cdot d\xi \quad\ldots (265)$$

This is easily proved to be the only solution of (253) satisfying the initial conditions prescribed. The expression (265) represents the so-called *d'Alembert's form* of the solution.

B. The general wave equation in three dimensions.

The homogeneous equation looks:

$$\frac{\partial^2 u}{\partial t^2}=a^2\cdot\triangle u, \quad\ldots\ldots\ldots\ldots\ldots\ldots (266)$$

a being as before a velocity, and $\triangle$ denoting the Laplacian symbol. We will assume the initial conditions

$$(u)_{t=0}=F(x,y,z),\quad \left(\frac{\partial u}{\partial t}\right)_{t=0}=G(x,y,z). \quad\ldots\ldots\ldots (267)$$

Putting $u(x,y,z,t)=U(x,y,z;p)\cdot H(t)$, we arrive as above at the equality

$$[p^2\cdot U-p^2\cdot F(x,y,z)-p\cdot G(x,y,z)]\cdot H(t)=a^2\cdot\triangle U\cdot H(t),$$

which proves that the basic integrator $U(x,y,z;p)$ must satisfy the partial differential equation

$$\triangle U-\frac{p^2}{a^2}\cdot U=-\frac{p^2}{a^2}\cdot F(x,y,z)-\frac{p}{a^2}\cdot G(x,y,z). \quad\ldots\ldots (268)$$

Now, it is a well-known fact, that the equation

$$\triangle V - \frac{p^2}{a^2} \cdot V = -4\pi \cdot \varrho\,(x, y, z) \;\ldots\ldots\ldots\ldots \quad (269)$$

has particular integrals of the form

$$V\,(x, y, z) = \iiint \frac{\varrho\,(\xi, \eta, \zeta)}{r} \cdot e^{\pm \frac{p}{a} r} \cdot d\omega, \;\ldots\ldots\ldots \quad (270)$$

$d\omega$ being an element of volume, and the integration being extended over the whole space. (If the density of mass ϱ extends to infinity, it must be prescribed to decrease sufficiently rapidly with an increasing radius-vector r).

Hence, in our case (268):

$$\varrho\,(x, y, z) = \frac{1}{4\pi} \cdot \frac{p^2}{a^2} \cdot F\,(x, y, z) + \frac{1}{4\pi} \cdot \frac{p}{a^2} \cdot G\,(x, y, z). \quad \ldots \quad (271)$$

For interpreting a volume integral of the type

$$v\,(x, y, z, t) = \iiint \frac{\varrho\,(\xi, \eta, \zeta)}{r} \cdot e^{\pm \frac{p}{a} r} \cdot H\,(t) \cdot d\omega \;\ldots\ldots \quad (272)$$

we conveniently employ polar coordinates (r, θ, φ). The pole being situated at (x, y, z) we have the relations:

$$\left.\begin{aligned} \xi &= x + r \cdot \sin\theta \cdot \cos\varphi, \\ \eta &= y + r \cdot \sin\theta \cdot \sin\varphi, \\ \zeta &= z + r \cdot \cos\theta, \\ d\omega &= r^2 \cdot \sin\theta \cdot dr \cdot d\theta \cdot d\varphi. \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots \quad (273)$$

Accordingly,

$$v\,(x, y, z, t) = \int_0^\infty \int_0^\pi \int_0^{2\pi} \varrho\,(x + r \cdot \sin\theta \cdot \cos\varphi,\, y + r \cdot \sin\theta \cdot \sin\varphi,\, z + r \cdot \cos\theta) \cdot H\left(t \pm \frac{r}{a}\right) \cdot r \cdot \sin\theta \cdot dr \cdot d\theta \cdot d\varphi. \;\ldots\ldots\ldots\ldots \quad (274)$$

The factor $H\left(t \pm \frac{r}{a}\right)$ differs from zero and equals 1 only when $at \pm r > 0$, i. e. when $\mp r < at$. Now, at and r being positive, the condition $-r < at$ is *eo ipso* satisfied. The second condition,

$r < at$, brings a limitation as to the variation of r. Obviously, instead of integrating with respect to r between the limits 0 and ∞ we may as well integrate from 0 to at, thus obtaining

$$v(x, y, z, t) = \int_0^{at} dr \int_0^{\pi}\int_0^{2\pi} r\cdot\varrho\,(x+r\cdot\sin\theta\cdot\cos\varphi,\ y+r\cdot\sin\theta\cdot\sin\varphi,\ z+r\cdot\cos\theta)\cdot \sin\theta\cdot d\theta\cdot d\varphi \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (275)$$

In the expression (271) also a factor p (or p^2) equivalent to a differentiation symbol occurs. By (275)

$$p\cdot v(x, y, z, t) = \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (276)$$
$$a^2\cdot t\cdot \int_0^{\pi}\int_0^{2\pi} \varrho\,(x+at\cdot\sin\theta\cdot\cos\varphi,\ y+at\cdot\sin\theta\cdot\sin\varphi,\ z+at\cdot\cos\theta)\cdot \cdot\sin\theta\cdot d\theta\cdot d\varphi.$$

Hence our solution of (266) will appear in the form

$$u(x, y, z, t) = \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (277)$$
$$\frac{\partial}{\partial t}\left[\frac{t}{4\pi}\int_0^{\pi}\int_0^{2\pi} F\,(x+at\cdot\sin\theta\cdot\cos\varphi,\ y+at\cdot\sin\theta\cdot\sin\varphi,\ z+at\cdot\cos\theta)\cdot \cdot\sin\theta\cdot d\theta\cdot d\varphi\right]$$
$$+\frac{t}{4\pi}\cdot\int_0^{\pi}\int_0^{2\pi} G\,(x+at\cdot\sin\theta\cdot\cos\varphi,\ y+at\cdot\sin\theta\cdot\sin\varphi,\ z+at\cdot\cos\theta)\cdot \cdot\sin\theta\cdot d\theta\cdot d\varphi,$$

or

$$u(x, y, z, t) = \frac{\partial}{\partial t}[t\cdot\bar{F}(at)] + t\cdot\bar{G}(at), \quad \ldots\ldots \quad (277\text{ a})$$

if we denote by $\bar{F}$ (at) the *mean value* of the function $F(x, y, z)$ upon a sphere with radius at about (x, y, z).

Thus we have arrived at a well-known form of the solution of our problem and may refer to textbooks of mechanics as to the discussion of its physical significance.

In the case of an *inhomogeneous* wave-equation, which may be written

$$\frac{1}{a^2}\cdot\frac{\partial^2 u}{\partial t^2} = \triangle\, u + 4\pi\varrho\,(x, y, z, t), \quad \ldots\ldots\ldots \quad (266\text{ a})$$

we get instead of (268) the integrational equation

$$\Delta U - \frac{p^2}{a^2}\cdot U = -\frac{p^2}{a^2}\cdot F(x, y, z) - \frac{p}{a^2}\cdot G(x, y, z) - 4\pi\cdot R(x, y, z; p), \quad (268\,\mathrm{a})$$

$R(x, y, z; p)$ denoting the basic integrator of the density function $\varrho(x, y, z, t)$. Accordingly, to the solution of the homogeneous equation just obtained we now have to add the term

$$u_1(x, y, z, t) = \iiint \frac{R(\xi, \eta, \zeta; p)}{r}\cdot e^{\pm\frac{p}{a}r}\cdot H(t)\cdot d\omega. \;\ldots\ldots\ldots \quad (278)$$

But here

$$R(\xi, \eta, \zeta;\; p)\cdot e^{\pm\frac{p}{a}r}\cdot H(t) = e^{\pm\frac{p}{a}r}\cdot\varrho(\xi, \eta, \zeta, t) = \varrho\left(\xi, \eta, \zeta, t \pm \frac{r}{a}\right)\cdot$$

Now we will make the assumption that also the volume density, being a function of time, keeps a constant value until the moment $t=0$. In that case the solution corresponding to the + sign obviously must be omitted, and we get

$$u_1(x, y, z, t) = \iiint \frac{1}{r}\cdot\varrho\left(\xi, \eta, \zeta, t - \frac{r}{a}\right)\cdot d\omega. \;\ldots\ldots \quad (279)$$

Here the integral expresses the so-called *retarded potential* at the point (x, y, z) under consideration.

C. The equation of conduction of heat in three dimensions.

If we denote by a a constant of the dimension $L\cdot T^{-\frac{1}{2}}$, the homogeneous differential equation in question has the appearance

$$\frac{\partial u}{\partial t} = a^2\cdot\Delta u. \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (280)$$

We shall try to find a solution that for $t=0$ coincides with a given function of space, say

$$(u)_{t=0} = F(x, y, z). \;\ldots\ldots\ldots\ldots\ldots\ldots \quad (281)$$

Introducing $u(x, y, z, t) = U(x, y, z;\; p)\cdot H(t)$, one gets the following equation for the basic integrator

$$\Delta U - \frac{p}{a^2}\cdot U = -\frac{p}{a^2}\cdot F(x, y, z) \;\ldots\ldots\ldots\ldots \quad (282)$$

According to (269) and (270) this equation has particular solutions of the form

$$U(x, y, z; p) = \frac{p}{a^2}\cdot\frac{1}{4\pi}\iiint\frac{F(\xi, \eta, \zeta)}{r}\cdot e^{\pm\frac{r}{a}\sqrt{p}}\cdot d\omega \quad .. \quad (283)$$

We will employ the notation:

$$\varphi(r, t) = e^{\pm\frac{r}{a}\sqrt{p}}\cdot H(t), \ldots\ldots\ldots\ldots\ldots\ldots \quad (284)$$

thus obtaining, when $t > 0$,

$$u(x, y, z, t) = \frac{1}{4\pi a^2}\iiint\frac{F(\xi, \eta, \zeta)}{r}\cdot\varphi_t'(r, t)\cdot d\omega. \quad \ldots. \quad (285)$$

The basic integrator appearing in (284) is of a type already treated in the foregoing chapter. Thus, by formula (218)

$$\varphi(r, t) = \left[1 - \mathrm{erf}\left(\frac{r}{2a\sqrt{t}}\right)\right]\cdot H(t) \quad \ldots\ldots\ldots\ldots \quad (286)$$

for the applicable branch of the square root, and accordingly, when $t > 0$:

$$\varphi_t'(r, t) = \frac{1}{2\sqrt{\pi}}\cdot\frac{r}{a}\cdot\frac{1}{t\sqrt{t}}\cdot e^{-\frac{r^2}{4a^2t}} \ldots\ldots\ldots\ldots \quad (287)$$

Therefore, in agreement with (285), the solution of our problem must be

$$u(x, y, z, t) = \frac{1}{8\pi\sqrt{\pi}}\cdot\frac{1}{a^3}\cdot\frac{1}{t\sqrt{t}}\cdot\iiint F(\xi, \eta, \zeta)\cdot e^{-\frac{r^2}{4a^2t}}\cdot d\omega \quad .. \quad (288)$$

or, if we again employ polar coordinates (r, Θ, φ) and, to simplify our mode of writing, introduce the mean value of F

$$\bar{F}(r) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} F(\xi, \eta, \zeta)\cdot\sin\Theta\cdot d\Theta\cdot d\varphi \ldots\ldots \quad (289)$$

upon a sphere of radius r about $(x y z)$:

$$u(x, y, z, t) = \frac{1}{2\sqrt{\pi}}\cdot\frac{1}{(a\sqrt{t})^3}\cdot\int_0^{\infty}\bar{F}(r)\cdot e^{-\frac{r^2}{4a^2t}}\cdot r^2\cdot dr. \quad \ldots. \quad (290)$$

But this is just the classical *diffusion-formula.*

If we write the inhomogeneous equation of conduction of heat

$$\frac{\partial u}{\partial t}=a^2\cdot\Delta u+4\pi\varrho\,(x, y, z, t), \;\ldots\ldots\ldots\ldots \quad (280\,\mathrm{a})$$

we get in this case, instead of (282), the integrational equation

$$\Delta U-\frac{p}{a^2}\cdot U=-\frac{p}{a^2}\cdot F\,(x, y, z)-\frac{4\pi}{a^2}\cdot R\,(x, y, z;\, p), \ldots\ldots \quad (282\,\mathrm{a})$$

and accordingly we have to add to the above solution (283) a solution U_1 of the equation

$$\Delta U_1-\frac{p}{a^2}\cdot U_1=-\frac{4\pi}{a^2}\cdot R\,(x, y, z;\, p).$$

This solution is obtained from (270), giving

$$U_1\,(x, y, z;\, p)=\frac{1}{a^2}\iiint\frac{R\,(x, y, z;\, p)}{r}\cdot e^{\pm\frac{r}{a}\sqrt{p}}\cdot d\omega.$$

Attaching the unit-function $H\,(t)$ to both members, we get on account of (287), as $\varphi\,(r, 0)=0$, the following expression for the complementary function

$$u_1\,(x, y, z, t)=$$

$$\frac{1}{2\sqrt{\pi}}\iiint\left[\int_0^t e^{-\frac{r^3}{4a^2\tau}}\cdot\varrho\,(x, y, z, t-\tau)\cdot\frac{d\tau}{(a\sqrt{\tau})^3}\right]\cdot d\omega \ldots \quad (291)$$

Here, if suitable, the integrations with respect to ω and τ may generally be interchanged.

D. The universal homogeneous equation of telegraphy.

In the universal form of the wave equation time derivatives of the first as well as of the second order appear. In the one-dimensional case the equation of telegraphy offers a practically important example of such an equation. According to Chapter 6, formula (94), where we shall put B and E equal to zero, the voltage as well as the current is governed by the equation:

$$l\,c\cdot\frac{\partial^2 v}{\partial t^2}+(r\,c+a\,l)\cdot\frac{\partial v}{\partial t}+r\,a\cdot v=\frac{\partial^2 v}{\partial x^2}\cdot \;\ldots\ldots\ldots\ldots \quad (292)$$

We will now integrate this equation under the initial conditions

$$v\,(x,0)=F\,(x),\ \frac{\partial v}{\partial t}(x,0)=G\,(x). \qquad (293)$$

If we put $v\,(x,t)=V(x;p)\cdot H\,(t)$ and introduce $\delta\,(p)$, defined by (99), we obtain:

$$\frac{d^2V}{dx^2}-\delta^2(p)\cdot V=-p\,[p\cdot l\,c\cdot F\,(x)+(a\,l+r\,c)\cdot F\,(x)+l\,c\cdot G\,(x)]. \qquad (294)$$

Note. From an electrotechnical point of view it is more natural to assume the initial conditions

$$v\,(x,0)=F\,(x),\ i\,(x,0)=K\,(x). \qquad (293\,\mathrm{a})$$

Then the integrator equation looks

$$\frac{d^2V}{dx^2}-\delta^2(p)\cdot V=-p\ [c(r+pl)\cdot F\,(x)-l\cdot K'(x)] \qquad (294\,\mathrm{a})$$

This problem is therefore analytically equivalent to the former.

Accordingly, we have to treat an equation of the type:

$$\frac{d^2V}{dx^2}-\delta^2(p)\cdot V=p\cdot f\,(x). \qquad (295)$$

Making the attempt

$$V(x;p)=A\,(x)\cdot e^{\delta x}+B\,(x)\cdot e^{-\delta x},$$

we get, employing the method of varying the constants,

$$A'(x)=\frac{p}{2\delta}\cdot e^{-\delta x}\cdot f\,(x),\ B'\,(x)=-\frac{p}{2\delta}\cdot e^{\delta x}\cdot f\,(x)$$

and from this:

$$V\,(x;\,p)=\frac{p}{2\delta}\cdot\int\limits_{\alpha}^{x}e^{\delta(x-\xi)}\cdot f\,(\xi)\cdot d\xi-\frac{p}{2\delta}\cdot\int\limits_{\beta}^{x}e^{-\delta(x-\xi)}\cdot f\,(\xi)\cdot d\xi. \qquad (296)$$

Accordingly, the problem under consideration is essentially solved, if we know how to interpret the special integrational expression $\frac{p}{\delta(p)}\cdot e^{-\zeta\cdot\delta(p)}$ $(\zeta>0)$. We have already encountered this expression in Chapter 6, and now we will determine the corresponding primary function

$$\chi\,(\zeta,t)=\frac{p}{\delta(p)}\cdot e^{-\zeta\cdot\delta(p)}\cdot H\,(t). \qquad (297)$$

For that scope we conveniently rewrite the transmission exponent $\delta(p)$ under the form

$$\delta(p)=\frac{1}{u}\cdot\sqrt{(p+\varkappa)^2-\lambda^2}, \ldots\ldots\ldots\ldots\ldots \quad (298)$$

where

$$\varkappa=\frac{1}{2}\left(\frac{r}{l}+\frac{a}{c}\right),\quad \lambda=\frac{1}{2}\left(\frac{r}{l}-\frac{a}{c}\right),\quad u=\frac{1}{\sqrt{l\,c}}\cdot \ldots\ldots\ldots \quad (299)$$

Hence we can write $\chi(\zeta, t)$ more completely

$$\chi(\zeta,t)=u\cdot\frac{p}{\sqrt{(p+\varkappa)^2-\lambda^2}}\cdot e^{-\frac{\zeta}{u}\cdot\sqrt{(p+\varkappa)^2-\lambda^2}}\cdot H(t). \quad \ldots\ldots \quad (300)$$

We may easily deduce the explicit function by making use of the following integrational rule, which has been proved in the foregoing chapter:

$$\frac{p}{\sqrt{p^2+\omega^2}}\cdot e^{-a\sqrt{p^2+\omega^2}}\cdot H(t)=J_0\left(\omega\cdot\sqrt{t^2-a^2}\right)\cdot H(t-a). \ldots \quad (301)$$

If we put $\omega=i\lambda$ and employ the common notation $J_0(i\,x)=I_0(x)$, we get another appearance of this rule, viz

$$\frac{p}{\sqrt{p^2-\lambda^2}}\cdot e^{-a\sqrt{p^2-\lambda^2}}\cdot H(t)=I_0\left(\lambda\cdot\sqrt{t^2-a^2}\right)\cdot H(t-a) \ldots \quad (301\,a)$$

To make this formula applicable to our problem we must at first somewhat transform (300) according to the so-called displacement rule

$$\Phi(p+\varkappa)\cdot f(t)=e^{-\varkappa t}\cdot\Phi(p)\cdot e^{\varkappa t} f(t).$$

Thus we get, writing

$$\chi(\zeta,t)=u\cdot\frac{p+\varkappa}{\sqrt{(p+\varkappa)^2-\lambda^2}}\cdot e^{-\frac{\zeta}{u}\sqrt{(p+\varkappa)^2-\lambda^2}}\cdot\frac{p}{p+\varkappa}\cdot H(t),$$

because

$$\frac{p}{p+\varkappa}\cdot H(t)=e^{-\varkappa t}\cdot H(t),$$

$$\chi(\zeta,t)=u\cdot e^{-\varkappa t}\cdot\frac{p}{\sqrt{p^2-\lambda^2}}\cdot e^{-\frac{\zeta}{u}\sqrt{p^2-\lambda^2}}\cdot H(t). \quad \ldots\ldots \quad (302)$$

Now (301 a) can be directly applied, and we obtain $\chi(\zeta, t)$ under the form

$$\chi(\zeta, t)=u\cdot e^{-\varkappa t}\cdot I_0\left(\lambda\cdot\sqrt{t^2-\left(\frac{\zeta}{u}\right)^2}\right)\cdot H\left(t-\frac{\zeta}{u}\right) \quad \ldots\ldots \quad (303)$$

After that we will return to formula (296) and make both members operate on $H(t)$, thus getting

$$v(x, t)=-\frac{1}{2}\int_x^{\alpha}\chi(\xi-x, t)\cdot f(\xi)\cdot d\xi-\frac{1}{2}\int_{\beta}^{x}\chi(x-\xi, t)\cdot f(\xi)\cdot d\xi \ldots \quad (304)$$

Now, by (303), $\chi(\xi-x, t)\equiv 0$ if $\xi>x+ut$, and accordingly the limits of the first integral may be as well x and $x+ut$. Further $\chi(x-\xi, t)\equiv 0$ if $\xi<x-ut$, and thus we can take the limits $x-ut$ and x in the last integral. When the function is not zero, we have also $\chi(-\zeta, t)=\chi(\zeta, t)$. Therefore the last equality (304) may be written

$$v(x, t)=-\frac{1}{2}\cdot\int_{x-ut}^{x+ut}\chi(x-\xi, t)\cdot f(\xi)\cdot d\xi \quad \ldots\ldots\ldots\ldots \quad (305)$$

From this we obtain by well-known methods

$$p\cdot v(x, t)=\frac{\partial v(x, t)}{\partial t}=$$

$$-\frac{u}{2}\cdot[\chi(-ut, t)\cdot f(x+ut)+\chi(ut, t)\cdot f(x-ut)] \quad \ldots\ldots \quad (306)$$

$$-\frac{1}{2}\cdot\int_{x-ut}^{x+ut}f(\xi)\cdot\frac{\partial}{\partial t}\chi(x-\xi, t)\cdot d\xi.$$

Further it follows from (303), when $\zeta=-ut+0$ or $\zeta=ut-0$, that

$$\chi(\pm ut, t)=u\cdot e^{-\varkappa t}\cdot I_0(0)\cdot H(arg\geq+0)=u\cdot e^{-\varkappa t}, \quad \ldots\ldots \quad (307)$$

and from (305) and (306) we get

$$v(x, t)=-\frac{u}{2}\cdot e^{-\varkappa t}\cdot\int_{x-ut}^{x+ut}f(\xi)\cdot I_0\left[\lambda\cdot\sqrt{t^2-\left(\frac{x-\xi}{u}\right)^2}\right]\cdot d\xi, \quad \ldots\ldots \quad (308)$$

$$\frac{\partial v(x,t)}{\partial t}=-\frac{u^2}{2}\cdot e^{-\varkappa t}\cdot[f(x+ut)+f(x-ut)]$$

$$+\frac{u\cdot\varkappa}{2}\cdot e^{-\varkappa t}\cdot\int\limits_{x-ut}^{x+ut} f(\xi)\cdot I_0\left[\lambda\cdot\sqrt{t^2-\left(\frac{x-\xi}{u}\right)^2}\,\right]\cdot d\xi \quad \ldots\ldots \quad (309)$$

$$-\frac{u}{2}\cdot e^{-\varkappa t}\cdot\int\limits_{x-ut}^{x+ut} f(\xi)\cdot\frac{\partial}{\partial t}I_0\left[\lambda\cdot\sqrt{t^2-\left(\frac{x-\xi}{u}\right)^2}\,\right]\cdot d\xi.$$

Into the expression for $v(x,t)$ we shall now, according to (294), insert

$$-f(\xi)=lc\cdot p\cdot F(\xi)+(al+rc)\cdot F(\xi)+lc\cdot G(\xi) \quad \ldots\ldots \quad (310)$$

or, as follows from (299),

$$-f(\xi)=\frac{1}{u^2}\cdot[p\cdot F(\xi)+2\varkappa\cdot F(\xi)+G(\xi)]. \quad \ldots\ldots\ldots \quad (310\,\mathrm{a})$$

Hence formula (308) gets the appearance

$$v(x,t)=$$

$$\frac{e^{-\varkappa t}}{2u}\cdot\int\limits_{x-ut}^{x+ut}[2\varkappa\cdot F(\xi)+G(\xi)]\cdot I_0\left[\lambda\cdot\sqrt{t^2-\left(\frac{x-\xi}{u}\right)^2}\,\right]\cdot d\xi$$

$$+\frac{1}{u^2}\cdot\frac{\partial}{\partial t}\left\{\frac{u}{2}\cdot e^{-\varkappa t}\cdot\int\limits_{x-ut}^{x+ut}F(\xi)\cdot I_0\left[\lambda\cdot\sqrt{t^2-\left(\frac{x-\xi}{u}\right)^2}\,\right]\cdot d\xi.\right. \quad (311)$$

If we here compute the derivative according to (309), we arrive at the following ultimate expression for the voltage function:

$$v(x,t)=e^{-\varkappa t}\cdot\frac{1}{2}\cdot[F(x-ut)+F(x+ut)]$$

$$+\frac{e^{-\varkappa t}}{2u}\cdot\int\limits_{x-ut}^{x+ut}[\varkappa\cdot F(\xi)+G(\xi)]\cdot I_0\left[\lambda\cdot\sqrt{t^2-\left(\frac{x-\xi}{u}\right)^2}\,\right]\cdot d\xi \quad (312)$$

$$+\frac{e^{-\varkappa t}}{2u}\cdot\int\limits_{x-ut}^{x+ut}F(\xi)\cdot\frac{\partial}{\partial t}I_0\left[\lambda\cdot\sqrt{t^2-\left(\frac{x-\xi}{u}\right)^2}\,\right]\cdot d\xi.$$

In this form, but deduced in another way, the solution of the problem is presented in Webster's textbook of partial differential equations. We observe that, besides the exponentially attenuated wave process given by the first term of (312), there are also present constituents of quite another nature. To the corresponding value of the function at any point x will contribute the whole set of values of the functions F and G within the interval $x \pm ut$. Hence we see that the procedure will not be like a wave travelling over the point, the effect of which may set in at one moment and cease again after some time, but will continue for ever after the moment of beginning. Also the latter terms of the expression are furnished with an exponential damping factor with respect to time.

E. A disturbance problem of the telephone technic.

As another application of a direct practical interest we will now treat a special case of the general transmission-line problem outlined in Chapter 6. Thus we take under consideration a homogeneous telephone line of length s, equipped with some apparatus at the ends, the line being suddenly exposed to the sinusoidal magnetic field of a parallel power transmission line. The disturbance field is supposed to be the same along the telephone line; it will cause a voltage transient on this, the nature of which we are going to examine.

The functions Φ and Ψ in (96) now get the very simple form

$$\left.\begin{aligned} \Phi(x;p) &= \Phi(p) = d \cdot p \cdot [B(p) - \mathrm{B}(0)] = \frac{\omega p^2}{p^2+\omega^2} \cdot d \cdot \mathrm{B}_0, \\ \Psi(x;p) &= 0. \end{aligned}\right\} \ldots \quad (313)$$

Then we have assumed that $\mathrm{B}(t) = \mathrm{B}_0 \cdot \sin \omega t \cdot H(t)$.

By (98) and (100) the voltage equation will be

$$\frac{d^2V}{dx^2} - \delta^2(p) \cdot V = 0.$$

In accordance with the methods of Chapter 6 we want a solution of this homogeneous equation under the inhomogeneous boundary conditions

$$\left.\begin{aligned} V(0;p)&=E'_*(p)+\frac{Z'(p)}{r+pl}\cdot\frac{dV}{dx}(0;p),\\ -V(s;p)&=E''_*(p)+\frac{Z''(p)}{r+pl}\cdot\frac{dV}{dx}(s;p),\end{aligned}\right\} \quad\ldots\ldots\ (314)$$

where

$$\left.\begin{aligned} E'_*(p)&=\frac{Z'(p)}{r+pl}\cdot\frac{\omega p^2}{p^2+\omega^2}\cdot d\cdot \mathrm{B}_0,\\ E''_*(p)&=\frac{Z''(p)}{r+pl}\cdot\frac{\omega p^2}{p^2+\omega^2}\cdot d\cdot \mathrm{B}_0.\end{aligned}\right\} \quad\ldots\ldots\ldots\ (314\,\mathrm{a})$$

No inhomogeneous differential equation occurs in this special case.

The complete solution of the problem we have already got expressed by the formulas (104 a) or (125). For the sake of simplicity we will confine ourselves to the simple case of a distortionless line, the ends being circuited by pure resistances. Then

$$\delta=\frac{1}{u}(p+\mu),\ Z(p)=R,\ Z'(p)=R',\ Z''(p)=R'' \quad\ldots.\ (315)$$

and, according to (121), the coefficients of reflection are pure numbers, in magnitude <1:

$$G'=\frac{R-R'}{R+R'},\ G''=\frac{R-R''}{R+R''}\cdot \quad\ldots\ldots\ldots\ldots\ldots\ (316)$$

Further, the electromotive forces are $\left(\mu=\frac{r}{l}\right)$:

$$\left.\begin{aligned} E'_*(p)&=\frac{p}{p+\mu}\cdot\frac{\omega p}{p^2+\omega^2}\cdot\frac{R'}{l}\cdot d\cdot \mathrm{B}_0\\ E''_*(p)&=\frac{p}{p+\mu}\cdot\frac{\omega p}{p^2+\omega^2}\cdot\frac{R''}{l}\cdot d\cdot \mathrm{B}_0.\end{aligned}\right\} \quad\ldots\ldots\ldots\ (317)$$

The corresponding primary functions may be written down immediately on account of the formulas (74) and (75). Hence we obtain, adding the factor $H(t)$ to each side,

$$e'_*(t)=\frac{\omega}{\omega^2+\mu^2}\cdot(-\mu\cdot e^{-\mu t}+\mu\cdot\cos\omega t+\omega\cdot\sin\omega t)\cdot\frac{R'}{l}\cdot d\cdot \mathrm{B}_0\cdot H(t),$$
$$e''_*(t)=\frac{\omega}{\omega^2+\mu^2}\cdot(-\mu\cdot e^{-\mu t}+\mu\cdot\cos\omega t+\omega\cdot\sin\omega t)\cdot\frac{R''}{l}\cdot d\cdot \mathrm{B}_0\cdot H(t). \qquad (318)$$

Now, by (125)

$$v'(x,t)=W'(x;p)\cdot e'_*(t) \text{ and } v''(x,t)=W''(x;p)\cdot e''_*(t), \;.. \quad (319)$$

where W' and W'' are certain series expressions formed by terms of the type

$$\frac{R}{R+R'}\cdot G'^k\cdot G''^k\cdot e^{-\delta\cdot(2ks+x)}.$$

Operating upon $e'_*(t)$, this integrator gives:

$$\frac{R}{R+R'}\cdot G'^k\cdot G''^k\cdot e^{-\frac{\mu}{u}(2ks+x)}\cdot e'_*\left(t-\frac{2ks+x}{u}\right). \;\ldots\ldots \quad (320)$$

Hence we obtain a system of waves, exponentially attenuated and travelling along the telephone line in both directions with the velocity u. A reduction of the amplitudes of the waves is also caused by reflection-losses at the ends. In practical applications the waves are often damped so rapidly that even a few terms offer a sufficiently good picture of the transient disturbance process. The steady state disturbance is easily obtained by omitting the exponential term in the expressions for $e'_*(t)$ and $e''_*(t)$.

The general case, where the line is not distortionless and where Z' and Z'' are functions of p, can always be attacked formally in the same way, but we see at a glance that considerable complications will enter.

F. Application of integrational and operational methods to the double-pole-radiation problem.

The fundamental equations (Maxwell's equations) of the electromagnetic field may be written:

$$\left.\begin{aligned}\text{rot }\overline{H}&=4\pi\cdot\overline{I}+\frac{\partial\overline{D}}{\partial t},\\ \text{rot }\overline{E}&=-\frac{\partial\overline{B}}{\partial t}.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\quad(321)$$

$$\overline{B}=\mu\cdot\overline{H},\ \overline{D}=\varepsilon\cdot\overline{E},\ \overline{I}=\sigma\cdot(\overline{E}+\overline{E}_k).\quad\ldots\ldots\ldots\quad(322)$$

$\overline{H}$ denotes the vector of magnetic field strength, $\overline{E}$ that of electric field strength, $\overline{B}$ the magnetic induction, D the electric displacement, $\overline{I}$ the current density, and $\overline{E}_k$ the external electric field strength. μ is the permeability, ε the dielectric constant, and σ the conductivity.

We shall assume $\overline{E}_k$ to be zero and μ, ε, and σ to be constant all over the space. Further we prescribe the fields to be identically zero at the moment $t=0$. Thus, if henceforth we let the capital letters represent the *basic integrators,* we get the integrational equations:

$$\left.\begin{aligned}\text{rot }\overline{H}&=(4\pi\sigma+p\varepsilon)\cdot\overline{E},\\ \text{rot }\overline{E}&=-p\cdot\mu\cdot\overline{H}.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\quad(323)$$

From this the solenoidal nature of the vector fields necessarily follows, i. e.

$$\text{div }\overline{E}=0,\ \text{div }\overline{H}=0.\quad\ldots\ldots\ldots\ldots\ldots\quad(324)$$

We will examine the process of radiation from an electric double-pole, the axis of which we take as one axis of a spherical coordinate system (r, ϑ, φ). From the cylindrical symmetry with respect to this axis follows that all magnitudes will be independent of the angle φ, and thus we may put $\frac{\partial}{\partial\varphi}=0$ in the equations. In the very neighbourhood of the doublet, or the »rational current element» as it is sometimes called, the finite propagation-velocity of the disturbance can be let out of consideration, i. e. we may as a first approximation assume the fields to change simultaneously at all points and to coincide with the fields in the so-called *quasi-stationary* case. In this the components E_φ, H_r, and H_ϑ are all zero, which means that the electric vector lines are situated in planes through the axis, and the magnetic lines are circles with their centra upon the axis and their planes orthogonal to it. The magnitude of the vector $\overline{H}$, $H=H_{\varphi}$, is connected with the current $i=\frac{dq}{dt}$ in the element according to *Biot-Savart's formula*, which we may write in the integrational form

$$H=\frac{\sin\vartheta}{r^2}\cdot l\cdot p\cdot Q, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (325)$$

Q being the basic integrator of the charge and l the distance between the two poles.

We shall make tentatively the assumption that the general radiation field of the doublet has the same fundamental geometrical character as the one just outlined, and especially that, when the distance r tends to zero, the analytic form of the magnetic field approaches the expression (325). Then, according to well-known formulas as to the operations rot and div in spherical coordinates, we get from (323) and (324) the following system of partial differential equations:

$$\left.\begin{aligned}
&\frac{1}{r\cdot\sin\vartheta}\cdot\frac{\partial}{\partial\vartheta}(\sin\vartheta\cdot H)=(4\pi\sigma+p\varepsilon)\cdot E_r,\\
&-\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\cdot H)=(4\pi\sigma+p\varepsilon)\cdot E_{\vartheta},\\
&\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\cdot E_{\vartheta})-\frac{1}{r}\cdot\frac{\partial E_r}{\partial\vartheta}=-p\mu\cdot H,\\
&\sin\vartheta\cdot\frac{\partial}{\partial r}(r^2\cdot E_r)+r\cdot\frac{\partial}{\partial\vartheta}(\sin\vartheta\cdot E_{\vartheta})=0.
\end{aligned}\right\}\ldots\ldots \quad (326)$$

From the last equation we conclude that E_r and E_{ϑ} can be derived from an auxiliary function U. Indeed, it will be identically satisfied if we put:

$$\left.\begin{aligned}
E_r&=\frac{1}{r^2\cdot\sin\vartheta}\cdot\frac{\partial U}{\partial\vartheta},\\
E_{\vartheta}&=-\frac{1}{r\cdot\sin\vartheta}\cdot\frac{\partial U}{\partial r}.
\end{aligned}\right\}\ \ldots\ldots\ldots\ldots \quad (327)$$

Then, from the second equation (326):

$$-\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\cdot H)=-\frac{4\pi\sigma+p\varepsilon}{r\cdot\sin\vartheta}\cdot\frac{\partial U}{\partial r}.$$

Here we may divide by $\frac{1}{r}$ and then integrate both the members with respect to r, omitting the constant of integration (Cf. (326)). Thus

$$H=\frac{4\pi\sigma+p\varepsilon}{r\cdot\sin\vartheta}\cdot U. \quad \ldots\ldots\ldots\ldots\ldots \quad (328)$$

Inserting the expressions found for H and E_r into the first equation (326), one finds this to be identically satisfied. From the remaining equation of the system we get at last, putting

$$\delta^2(p)=p^2\cdot\varepsilon\mu+p\cdot4\pi\sigma\mu, \quad \ldots\ldots\ldots\ldots\ldots \quad (329)$$

a partial differential equation for U, viz

$$\frac{\partial^2 U}{\partial r^2}+\frac{\sin\vartheta}{r^2}\cdot\frac{\partial}{\partial\vartheta}\left(\frac{1}{\sin\vartheta}\cdot\frac{\partial U}{\partial\vartheta}\right)-\delta^2\cdot U=0. \quad \ldots\ldots\ldots \quad (330)$$

By the substitution

$$U=\sin^2\vartheta\cdot\Phi(r) \quad \ldots\ldots\ldots\ldots\ldots \quad (331)$$

one can eliminate the variable ϑ from this equation, thus obtaining an ordinary differential equation for Φ

$$\frac{d^2\Phi}{dr^2}=\left(\delta^2+\frac{2}{r^2}\right)\cdot\Phi \ldots\ldots\ldots\ldots\ldots \quad (332)$$

The field components, according to (327), (328), (329), and (331), are now given by:

$$\left.\begin{aligned} H&=\frac{\delta^2}{p\mu}\cdot\sin\vartheta\cdot\frac{\Phi}{r},\\ E_r&=2\cdot\cos\vartheta\cdot\frac{\Phi}{r^2},\\ E_\vartheta&=-\sin\vartheta\cdot\frac{\Phi'}{r}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (333)$$

Two independent particular integrals of (332) are known to be

$$\Phi_1 = A_1\cdot\left(\frac{1}{r}+\delta\right)\cdot e^{-\delta r},\quad \Phi_2 = A_2\cdot\left(\frac{1}{r}-\delta\right)\cdot e^{\delta r}\ \ldots\ldots\ (334)$$

The factor A_1 being determined so that H satisfies the condition (325) at the origin, we find, corresponding to Φ_1:

$$\left.\begin{aligned} H &= \sin\vartheta\cdot\left(\frac{\delta}{r}+\frac{1}{r^2}\right)\cdot e^{-\delta r}\cdot l\cdot p\cdot Q, \\ E_r &= \frac{2\mu p}{\delta^2}\cdot\cos\vartheta\cdot\left(\frac{\delta}{r^2}+\frac{1}{r^3}\right)\cdot e^{-\delta r}\cdot l\cdot p\cdot Q, \\ E_\vartheta &= \frac{\mu p}{\delta^2}\sin\vartheta\cdot\left(\frac{\delta^2}{r}+\frac{\delta}{r^2}+\frac{1}{r^3}\right)\cdot e^{-\delta r}\cdot l\cdot p\cdot Q. \end{aligned}\right\}\ \ldots\ldots\ (335)$$

Replacing δ by $-\delta$, we get the system corresponding to Φ_2.

In the most important case the surrounding medium is an isolator, i. e. $\sigma=0$. Then, from (329), if we choose a suitable branch of the square root:

$$\delta = \frac{p}{u},\quad u = \frac{1}{\sqrt{\varepsilon\mu}}\cdot\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (336)$$

In this case we have the interpretation

$$e^{-\delta r}\cdot p\cdot Q\cdot H(t) \doteqdot e^{-p\frac{r}{u}}\cdot q'(t) = q'\left(t-\frac{r}{u}\right),\ \ldots\ldots\ldots\ (337)$$

the right hand side representing a spherical wave travelling out from the doublet with the velocity u. The other branch of the root corresponds to a similar wave converging towards the origin; obviously such a wave cannot exist in our state of affairs. Accordingly, attaching the unit-function $H(t)$ to each member and interpreting p as the differentiation-symbol $\frac{\partial}{\partial t}$, and at last denoting for short by m the *moment* $l\cdot q\left(t-\frac{r}{u}\right)$ of the doublet at the instant $t-\frac{r}{u}$, we obtain the following explicit expressions for the field components:

$$\left.\begin{aligned}
H&=\sin\vartheta\cdot\left(\frac{m''}{ru}+\frac{m'}{r^2}\right),\\
E_r&=\cos\vartheta\cdot\left(\frac{2\mu u\cdot m'}{r^2}+\frac{2\mu u^2\cdot m}{r^3}\right),\\
E_\vartheta&=\sin\vartheta\cdot\left(\frac{\mu\cdot m''}{r}+\frac{\mu u\cdot m'}{r^2}+\frac{\mu u^2\cdot m}{r^3}\right).
\end{aligned}\right\}\quad\ldots\ldots(338)$$

Because, according to our assumptions, $q(t)=0$ when $t<0$, there is no electric or magnetic field strength present at a point at a distance r from the doublet, until the time $\frac{r}{u}$ has elapsed.

When the conductivity of the medium is not zero, the radiation problem becomes much more complicated. We shall now briefly outline how it is possible to attack this problem by integrational methods. One general way of treating a compound integrator is to split it into terms or factorials of simpler forms or of standard types the interpretation of which is already known. For that scope we first rewrite the expressions for the field components under the following form:

$$\left.\begin{aligned}
H&=l\cdot\sin\vartheta\cdot\left[\frac{1}{r}\cdot\delta^2\cdot Q\cdot\frac{p}{\delta}\cdot e^{-r\delta}\cdot H(t)\right.\\
&\qquad\left.+\frac{1}{r^2}\cdot\frac{\delta^2}{p}\cdot\frac{p}{\delta}\cdot Q\cdot\frac{p}{\delta}\cdot e^{-r\delta}\cdot H(t)\right],\\
E_r&=2\mu\cdot l\cdot\cos\vartheta\cdot\left[\frac{1}{r^2}\cdot p\cdot Q\cdot\frac{p}{\delta}\cdot e^{-r\delta}\cdot H(t)\right.\\
&\qquad\left.+\frac{1}{r^3}\cdot\frac{p}{\delta}\cdot Q\cdot\frac{p}{\delta}\cdot e^{-r\delta}\cdot H(t)\right],\\
E_\vartheta&=\mu\cdot l\cdot\sin\vartheta\cdot\left[\frac{1}{r}\cdot\delta^2\cdot\frac{p}{\delta}\cdot Q\cdot\frac{p}{\delta}\cdot e^{-r\delta}\cdot H(t)\right.\\
&\qquad\left.+\frac{1}{r^2}\cdot p\cdot Q\cdot\frac{p}{\delta}\cdot e^{-r\delta}\cdot H(t)+\frac{1}{r^3}\cdot\frac{p}{\delta}\cdot Q\cdot\frac{p}{\delta}\cdot e^{-r\delta}\cdot H(t)\right].
\end{aligned}\right\}\quad\ldots(339)$$

Here all the factors appearing are familiar integrators. Because of the formulas (297), (299), (303), and (329) we have:

$$\left.\begin{aligned} \chi(t) &\equiv \frac{p}{\delta}\cdot e^{-r\cdot\delta}\cdot H(t) = u\cdot e^{-\varkappa t}\cdot I_0\left(\lambda\cdot\sqrt{t^2-\left(\frac{r}{u}\right)^2}\right)\cdot H\left(t-\frac{r}{u}\right), \\ \delta(p) &= \frac{1}{u}\cdot\sqrt{(p+\varkappa)^2-\lambda^2}, \\ u &= \frac{1}{\sqrt{\varepsilon\mu}},\quad \varkappa=\lambda=\frac{2\pi\sigma}{\varepsilon}. \end{aligned}\right\} \qquad (340)$$

For the special value $r=0$ we get the interpretation of another integrational factor, $\frac{p}{\delta}$. Hence

$$\psi(t) \equiv \frac{p}{\delta}\cdot H(t) = u\cdot e^{-\varkappa t}\cdot I_0(\lambda t)\cdot H(t). \quad \ldots\ldots\ldots \quad (341)$$

At last we meet in (339), besides $Q(p)$, which we may regard as a given basic integrator, the factors δ^2, $\frac{\delta^2}{p}$, and p, symbolizing pure differentiation processes. Indeed, from (329) follows that:

$$\left.\begin{aligned} \delta^2 &= \varepsilon\mu\cdot\frac{\partial^2}{\partial t^2}+4\pi\sigma\mu\cdot\frac{\partial}{\partial t}, \\ \frac{\delta^2}{p} &= \varepsilon\mu\cdot\frac{\partial}{\partial t}+4\pi\sigma\mu. \end{aligned}\right| \quad \ldots\ldots\ldots\ldots \quad (342)$$

It may be sufficient to carry through the method upon one of the terms. We choose one of the most complicated, say the second term of the expression for the magnetic field strength:

$$H^* = \frac{l\cdot\sin\vartheta}{r^2}\cdot\frac{\delta^2}{p}\cdot\frac{p}{\delta}\cdot Q(p)\cdot\frac{p}{\delta}\cdot e^{-r\delta}\cdot H(t) \quad \ldots\ldots\ldots\ldots \quad (343)$$

If Q and Ψ denote the integral operators $(q(0)=0)$

$$Q=\left[\int_0^t q'(t-\tau)\cdot d\tau\right],\quad \Psi=\left[\psi(0)+\int_0^t \psi'(t-\tau)\cdot d\tau\right], \quad \ldots\ldots \quad (344)$$

this can be presented in the more explicit form

$$H^* = \frac{l\cdot\sin\vartheta}{r^2}\cdot\left(\varepsilon\mu\cdot\frac{\partial}{\partial t}+4\pi\sigma\mu\right)\cdot\Psi\times Q\times\chi(t). \quad \ldots\ldots \quad (345)$$

From (341) we see that $\psi(0)=u$, and for $t>0$

$$\psi'(t)=u\cdot e^{-\varkappa t}\cdot\{-\varkappa\cdot I_0(\lambda t)+\lambda\cdot I'_0(\lambda t)\}. \quad \ldots\ldots\ldots \quad (346)$$

Thus we know the expression for the operator Ψ.

If we put $v(t)=\Psi\times\chi(t)$, we get:

$$H^*=\frac{l\cdot\sin\vartheta}{r^2}\cdot\Big\{\varepsilon\mu\cdot q'(0)\cdot v(t)+$$

$$+4\pi\sigma\mu\cdot\int_0^t q'(t-\tau)\cdot v(\tau)\,d\tau+\varepsilon\mu\cdot\int_0^t q''(t-\tau)\cdot v(\tau)\,d\tau\Big\},$$

$$v(t)=u^2\cdot e^{-\varkappa t}\Big\{I_0\left[\lambda\cdot\sqrt{t^2-\left(\frac{r}{u}\right)^2}\right]+\int_{\frac{r}{u}}^{t}[-\varkappa\cdot I_0(\lambda\cdot(t-\tau))+ \qquad (347)$$

$$+\lambda\cdot I'_0(\lambda\cdot(t-\tau))]\cdot I_0\left[\lambda\cdot\sqrt{\tau^2-\left(\frac{r}{u}\right)^2}\right]\cdot d\tau\Big\}\cdot H\left(t-\frac{r}{u}\right).$$

Accordingly we get a wave with the propagation-velocity u and a spatial damping-factor $e^{-\varkappa\frac{r}{u}}$ of the face. The function $v(t)$ characterizes especially the mechanism of radiation.

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12. Some applications of integrational methods to linear integral equations.

In Chapter 6, when studying the universal equation of telegraphy, we arrived at a differential equation of the type:

$$\frac{d^2V}{dx^2}-\delta^2(p)\cdot V=-F(x;p). \quad \ldots\ldots\ldots\ldots \quad (348)$$

In order to find the solution of this equation corresponding to the homogeneous boundary conditions

$$V(0;p)=\frac{Z'(p)}{r+pl}\cdot\frac{dV}{dx}(0;p),-V(s;p)=\frac{Z''(p)}{r+pl}\cdot\frac{dV}{dx}(s;p). \ldots \quad (349)$$

we first tried to determine the so-called *Green function* $K(x,\xi;p)$ belonging to the homogeneous equation under the same boundary conditions; having found this, we could form the solution desired according to the formula

$$V(x;p)=\int_0^s K(x,\xi;p)\cdot F(\xi;p)\cdot d\xi. \qquad \dots\dots\dots \quad (350)$$

Another mode of proceeding will bring us into the domain of linear integral equations. By this we take under consideration the simpler differential equation

$$\frac{d^2V}{dx^2}=-U(x;p) \qquad \dots\dots\dots\dots\dots\dots \quad (351)$$

and try to solve it under the same boundary conditions as before. Denoting by $H(x,\xi;p)$ the Green function of the differential expression $\frac{d^2V}{dx^2}$, we can write the solution of (351)

$$V(x)=\int_0^s H(x,\xi;p)\cdot U(\xi;p)\cdot d\xi. \qquad \dots\dots\dots \quad (352)$$

We can apply this result to the original equation (348) by introducing into (351):

$$U(x;p)=F(x;p)-\delta^2(p)\cdot V(x).$$

Then the equality (352) will appear in the form of a *linear integral equation* of *Fredholm's type:*

$$V(x)+\delta^2(p)\cdot\int_0^s H(x,\xi;p)\cdot V(\xi)\cdot d\xi=R(x;p), \qquad \dots\dots \quad (353)$$

where

$$R(x;p)=\int_0^s H(x,\xi;p)\cdot F(\xi;p)\cdot d\xi. \qquad \dots\dots\dots \quad (354)$$

The kernel of this equation, $H(x,\xi;p)$, is symmetric in x and ξ, but it is real only provided that p is real. One finds without difficulty the expressions:

$$\left.\begin{aligned} H(x,\xi;p) &= \frac{[Z'(p)+(r+pl)\cdot x]\ [Z''(p)+(r+pl)\cdot(s-\xi)]}{[Z'(p)+Z''(p)+(r+pl)\cdot s]\cdot(r+pl)}, \\ &\qquad \text{if } x\leq\xi \\ H(x,\xi;p) &= \frac{[Z'(p)+(r+pl)\cdot\xi]\cdot[Z''(p)+(r+pl)\cdot(s-x)]}{[Z'(p)+Z''(p)+(r+pl)\cdot s]\cdot(r+pl)}, \\ &\qquad \text{if } x\geq\xi \end{aligned}\right\} \quad \ldots (355)$$

Therefore, if $Z'(p)$ and $Z''(p)$ are rational functions of p, this kernel is itself such a function, and accordingly it is of a simpler structure than the Green function $K(x, \xi; p)$, which involves transcendentals in p (Cf. (107)).

The treatment of the equation (353) may be executed in accordance with the general methods of the theory of integral equations. From the point of integrational calculus special complications will enter, because the kernel $H(x, \xi; p)$ contains the parameter p. In the following we will discuss more in detail a special case of practical importance where that is not so. We arrive at this case putting

$$Z'(p)=s'\cdot(r+pl), \quad Z''(p)=s''\cdot(r+pl),$$

s' and s'' being positive numbers of the dimension of a length. A transmission-line can always be so connected at its ends that this condition is fulfilled. Then we have simply

$$H(x,\xi)=\frac{(s'+x)\,(s''+s-\xi)}{s'+s''+s}, \quad H(x,\xi)=\frac{(s'+\xi)\,(s''+s-x)}{s'+s''+s} \quad \ldots (356)$$
$$\text{if } x\leq\xi \qquad\qquad\qquad \text{if } x\geq\xi$$

Now the integral equation (353) has a real symmetric kernel, and we know that the corresponding *homogeneous* equation

$$\varphi(x)=\lambda\cdot\int_0^s H(x,\xi)\cdot\varphi(\xi)\cdot d\xi \ldots\ldots\ldots\ldots\ (357)$$

has real *characteristic numbers,* which may be proved to be all simple and positive. We will denote them by ω_1^2, ω_2^2, ... and the corresponding *normal functions* by $\varphi_1(x)$, $\varphi_2(x)$ ....

According to the so-called *Schmidt's formula* we get the following expression for $V(x)$:

$$V(x)=R(x;p)-\sum_{n=1}^{\infty}\frac{\delta^2(p)}{\delta^2(p)+\omega_n^2}\cdot R_n(p)\cdot\varphi_n(x), \quad \ldots\ldots \quad (358)$$

putting:

$$R_n(p)=\int_0^s R(\xi;p)\cdot\varphi_n(\xi)\cdot d\xi. \quad \ldots\ldots\ldots\ldots \quad (359)$$

By the definition formula (354) we obtain, making use of the symmetry of $H(x,\xi)$ and of (357),

$$\left.\begin{aligned} R_n(p)&=\int_0^s F(\xi;p)\cdot d\xi\int_0^s H(\xi,\xi_1)\cdot\varphi_n(\xi_1)\cdot d\xi_1=\\ &=\frac{1}{\omega_n^2}\cdot\int_0^s F(\xi;p)\cdot\varphi_n(\xi)\cdot d\xi. \end{aligned}\right\} \quad \ldots \quad (360)$$

Thus $R_n(p)$ differs only by the factor $\frac{1}{\omega_n^2}$ from the *Fourier-coefficient* of the function $F(x;p)$ with respect to the system of normal functions $\varphi_n(x)$.

In our case we can easily get some knowledge of the characteristic numbers and the normal functions. Indeed, putting $F(x,p)=0$, we infer that the integral equation (357) is equivalent to the differential equation ($\lambda=\omega^2$)

$$\frac{d^2\varphi}{dx^2}+\omega^2\cdot\varphi=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (361)$$

combined with the boundary conditions in question, viz

$$\varphi(0)=s'\cdot\varphi'(0), \quad -\varphi(s)=s''\cdot\varphi'(s). \quad \ldots\ldots\ldots \quad (362)$$

Writing the solution of (361)

$$\varphi(x)=A\cdot\sin\omega x+B\cdot\cos\omega x,$$

we get, according to (362), the following system to determine the coefficients A and B:

$$\left.\begin{array}{c}(\sin \omega s+\omega s''\cdot\cos \omega s)\cdot A+(\cos \omega s-\omega s''\cdot\sin \omega s)\cdot B=0\\ \omega s'\cdot A+(-1)\cdot B=0.\end{array}\right\}$$

The determinantal equation of this system, i. e.

$$\operatorname{tg} \omega s=\frac{\omega(s'+s'')}{\omega^2\cdot s's''-1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (363)$$

gives us the possible values of ω or λ.

The equation being transcendental, the properties of its roots may be most easily overlooked by a geometrical discussion. It means no serious limitation, principally at least, if for that purpose we assume $s'=s''=s$. Putting $\omega s=x$, our equation becomes:

$$\operatorname{tg} x=\frac{2x}{x^2-1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (363\text{ a})$$

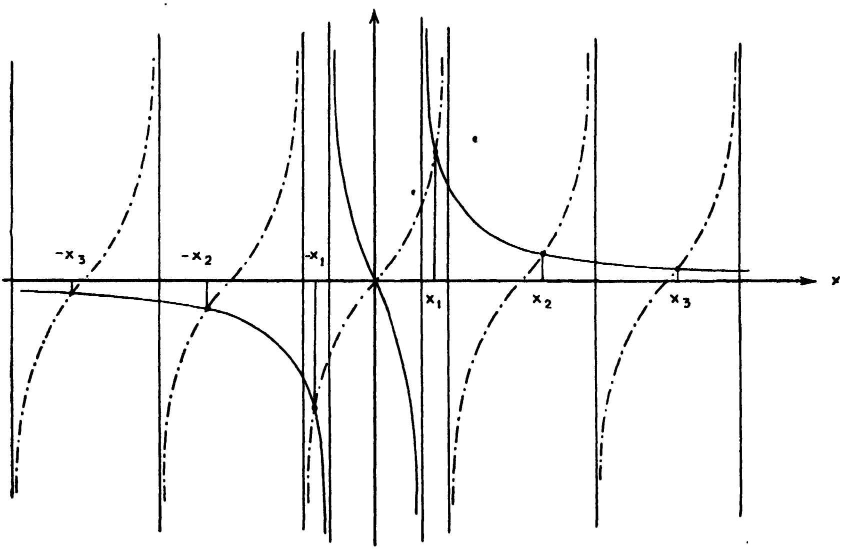

Fig. 10.

Referring to fig. 10, we note that $\lambda=\omega^2=\frac{x^2}{s^2}$ takes a sequence of positive values which asymptotically tend to the form

$$\lambda_n=\omega_n^2 \backsim n^2\cdot\frac{\pi^2}{s^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (364)$$

when n tends to infinity. Hence the normal functions are

$$\varphi_n(x)=A_n\cdot(\sin\omega_n x+\omega_n\cdot s'\cdot\cos\omega_n x), \quad \ldots\ldots\ldots \quad (365)$$

and A_n may be chosen so that $\varphi_n(x)$ is *normalized.* To each characteristic number ω_n^2 one normal function corresponds. The *orthogonality* of the normal functions can be verified e. g. from the symmetry of the function $H(x, \xi)$.

Now we return to the expression (358) for $V(x)$. Attaching the unit-function to both the members we obtain the time- and space-function $v(x; t)$ in the integrational form

$$v(x, t)=R(x; p)\cdot H(t)-\sum_{n=1}^{\infty}\varphi_n(x)\cdot\frac{\delta^2(p)}{\delta^2(p)+\omega_n^2}\cdot R_n(p)\cdot H(t), \quad \ldots \quad (366)$$

which shall now be interpreted.

At first we discuss the case where no electric or magnetic field acts upon the line, but there is originally a certain distribution of voltage $v(x, 0)$. Then, according to the definition formulas (100) an (96),

$$F(x; p)=(p^2\cdot lc+p\cdot rc)\cdot v(x, 0) \quad \ldots\ldots\ldots\ldots \quad (367)$$

and, according to (360),

$$R_n(p)=(p^2\cdot lc+p\cdot rc)\cdot\frac{1}{\omega_n^2}\int_0^s v(\xi; 0)\cdot\varphi_n(\xi)\cdot d\xi \quad \ldots\ldots \quad (368)$$
$$=(p^2\cdot lc+p\cdot rc)\cdot\frac{v_n}{\omega_n^2}\cdot$$

Further we obtain from (354):

$$R(x; p)\cdot H(t)=(p^2\cdot lc+p\cdot rc)\cdot\int_0^s H(x, \xi)\cdot v(\xi, 0)\cdot d\xi\cdot H(t)=0 \quad (369)$$

as, for $t>0$, p is equivalent to $\frac{d}{dt}\cdot$

Therefore

$$v(x,t)=-(p^2\cdot lc+p\cdot rc)\cdot\sum_{n=1}^{\infty}\frac{v_n}{\omega_n^2}\cdot\varphi_n(x)\cdot\frac{\delta^2(p)}{\delta^2(p)+\omega_n^2}\cdot H(t)\ \cdot\cdot\ \tag{370}$$

In this expression appears a function

$$w_n(t)=\frac{\delta^2(p)}{\delta^2(p)+\omega_n^2}\cdot H(t),\ \ldots\ldots\ldots\ldots\ldots \tag{371}$$

which obviously may be regarded as characteristic of the problem. In the universal case we have (Cf (299))

$$\delta^2(p)=\frac{1}{u^2}\cdot[(p+\kappa)^2-\lambda^2]$$

and thus

$$w_n(t)=\left\{1-u^2\cdot\omega_n^2\cdot\frac{1}{(p+\kappa)^2+\overline{\omega}_n^2}\right\}\cdot H(t)\ \ldots\ldots \tag{371 a}$$

if we put

$$\overline{\omega}_n^2=u^2\cdot\omega_n^2-\lambda^2.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{372}$$

Employing the expansion theorem, we readily arrive at the identity:

$$\begin{aligned}&\frac{1}{(p+\kappa)^2+\overline{\omega}_n^2}\cdot H(t)=\\&\frac{1}{\overline{\omega}_n^2+\kappa^2}\cdot\left[1-\frac{e^{-\kappa t}}{\overline{\omega}_n}\cdot(\overline{\omega}_n\cdot\cos\overline{\omega}_n t+\kappa\cdot\sin\overline{\omega}_n t)\right]\cdot H(t),\end{aligned}\tag{373}$$

from which for positive t we get the derivative

$$w'_n(t)=-u^2\cdot\frac{\omega_n^2}{\overline{\omega}_n}\cdot e^{-\kappa t}\cdot\sin\overline{\omega}_n t.$$

Accordingly, as $w_n(0)=1$, the integral operator corresponding to $w_n(t)$ must be:

$$W_n=\left[1-u^2\cdot\frac{\omega_n^2}{\overline{\omega}_n}\int_0^t e^{-\kappa(t-\tau)}\cdot\sin\overline{\omega}_n(t-\tau)\cdot d\tau\right]\ldots\ldots\ldots \tag{374}$$

We will soon make use of this result.

The general term of (370) involves the integrator

$$lc\cdot(p^2+p\mu)\cdot\frac{(p+\varkappa)^2-\lambda^2}{(p+\varkappa)^2+\bar{\omega}_n^2}, \quad \left(\mu=\frac{r}{l}\right)$$

(Cf. the definitions of λ and $\varkappa$ in formula (299)), the primary function of which may be obtained from (373) or directly by the expansion theorem. Thus we get from (366) the result

$$v(x,t)=\sum_{n=1}^{\infty} v_n\cdot\varphi_n(x)\cdot e^{-\varkappa t}\cdot\left(\cos\bar{\omega}_n t+\frac{\lambda}{\bar{\omega}_n}\cdot\sin\bar{\omega}_n t\right)\cdot H(t) \ldots \quad (375)$$

If we define an angle α according to the equations

$$\lambda=u\cdot\omega_n\cdot\cos\alpha,\quad \bar{\omega}_n=u\cdot\omega_n\cdot\sin\alpha,$$

our expression for the voltage can also be written:

$$v(x,t)=\sum_{n=1}^{\infty} u\cdot\frac{\omega_n}{\bar{\omega}_n}\cdot v_n\cdot\varphi_n(x)\cdot e^{-\varkappa t}\cdot\sin(\bar{\omega}_n t+\alpha)\cdot H(t). \ldots\ldots \quad (376)$$

When $t=0$, as is most easily seen from (375), this expression is reduced to:

$$v(x,0)=\sum_{n=1}^{\infty} v_n\cdot\varphi_n(x), \quad \ldots\ldots\ldots\ldots\ldots \quad (377)$$

which is the Fourier-development of $v(x,0)$ with respect to the system of normal functions $\varphi_1(x)$, $\varphi_2(x)$, ...

Now we will pass over to a brief examination of the case where the voltage as well as the current is initially zero, while the line at the moment $t=0$ is exposed to e. g. a magnetic disturbance. Then, according to (100) and (96),

$$F(x;p)=d\cdot p\cdot[B'_x(x;p)-\mathrm{B}'_x(x,0)]. \ldots\ldots\ldots \quad (378)$$

We will limit ourselves to the practically most common case, where the external magnetic field can be regarded as varying simultaneously along the line. Then we may put

$$\mathrm{B}(x,t)=\mathrm{B}(x)\cdot a(t)=\mathrm{B}(x)\cdot A(p)\cdot H(t) \ldots\ldots\ldots \quad (379)$$

and

$$F(x;\,p)=d\cdot p\cdot \mathrm{B}'(x)\,\left[A(p)-a(0)\right]. \quad\ldots\ldots\ldots\ldots \quad (380)$$

Further we have by (360)

$$R_n(p)=\frac{1}{\omega_n^2}\cdot d\cdot p\cdot\left[A(p)-a(0)\right]\cdot \mathrm{B}'_n, \quad\ldots\ldots\ldots\ldots \quad (381)$$

employing the notation

$$\mathrm{B}'_n=\int_0^s \mathrm{B}'(\xi)\cdot\varphi_n(\xi)\cdot d\xi, \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (382)$$

and from (354) follows:

$$R(x;\,p)=d\cdot p\cdot\left[A(p)-a(0)\right]\cdot\int_0^s H(x,\,\xi)\cdot \mathrm{B}'(\xi)\cdot d\xi. \quad\ldots\ldots \quad (383)$$

Thus we obtain, making both members operate on $H(t)$,

$$R_n(p)\cdot H(t)=\frac{d}{\omega_n^2}\cdot \mathrm{B}'_n\cdot a'(t), \quad\ldots\ldots\ldots\ldots \quad (384)$$

$$R(x;\,p)\cdot H(t)=d\cdot\int_0^s H(x,\,\xi)\cdot \mathrm{B}'(\xi)\cdot d\xi\cdot a'(t). \quad\ldots\ldots \quad (385)$$

Inserting these results into (366), we get the following expression for the voltage:

$$v(x,\,t)=$$

$$d\cdot\int_0^s H(x,\,\xi)\cdot \mathrm{B}'(\xi)\cdot d\xi\cdot a'(t)-d\cdot\sum_{n=1}^{\infty}\frac{\mathrm{B}'_n}{\omega_n^2}\cdot\varphi_n(x)\cdot\frac{\delta^2(p)}{\delta^2(p)+\omega_n^2}\cdot a'(t), \quad (386)$$

and can now immediately apply our formula (374) to arrive at the wanted explicit form of the function $v(x,\,t)$, which becomes:

$$v(x,t)=$$
$$=e^{-\varkappa t}\cdot u^2\cdot d\cdot\sum_{n=1}^{\infty}\frac{B'_n}{\bar{\omega}_n}\cdot\varphi_n(x)\cdot\int_0^t e^{\varkappa\tau}\cdot\sin\bar{\omega}_n(t-\tau)\cdot a'(\tau)\cdot d\tau, \tag{387}$$

as we have, because of the integral form (354) of *R* (*x; p*),

$$\int_0^s H(x,\xi)\cdot B'(\xi)\cdot d\xi=\sum_{n=1}^{\infty}\frac{B'_n}{\omega_n^2}\cdot\varphi_n(x).$$

In passing we remark that the function $v(x,t)$ derived above only represents one constituent of the real voltage transient. In fact, because of formula (103) we have another constituent obtained from the homogeneous differential equation under inhomogeneous boundary conditions. The examination of this part does not lead to integral equations and may be performed according to methods exemplified in the preceding chapter.

As a last example of the treatment by integrational methods of a problem where initial and boundary conditions are combined, we shall derive the analytic expression for the motion of an inhomogeneous elastic string with fixed ends.

The differential equation for the elongation $v(x,t)$ will then be of the form

$$\frac{\partial}{\partial x}\left(\alpha(x)\cdot\frac{\partial v}{\partial x}\right)=a(x)\cdot\frac{\partial^2 v}{\partial t^2}+e(x,t), \quad\ldots\ldots\ldots \tag{388}$$

where $\alpha(x)$ denotes the modulus of elasticity multiplied by the section, $a(x)$ the mass pro unit length, and $e(x,t)$ some function of disturbance.

Our mode of reasoning may as well be applied to the more general equation

$$\frac{\partial}{\partial x}\left(\alpha(x)\cdot\frac{\partial v}{\partial x}\right)+\beta(x)\cdot v=a(x)\cdot\frac{\partial^2 v}{\partial t^2}+e(x,t). \quad\ldots\ldots \tag{389}$$

Here the left hand side represents the most general *self-adjungated* linear differential form of the second order. The functions $\alpha(x)$, $\alpha'(x)$, and $\beta(x)$ are prescribed to be continuous in the interval $0\leq x\leq s$. We assume the initial conditions

$$v(x,0)=F(x), \quad \frac{\partial v}{\partial t}(x,0)=G(x) \quad \ldots\ldots\ldots \quad (390)$$

and the boundary conditions

$$v(0,t)=0, \quad v(s,t)=0. \quad \ldots\ldots\ldots\ldots \quad (391)$$

Then, of course, we must have $F(0)=F(s)=0$.

From a purely methodical point of view we could as well have chosen other homogeneous boundary conditions.

Now we introduce the basic integrators of $v(x,t)$ and $e(x,t)$, putting

$$v(x,t)=V(x;p)\cdot H(t), \quad e(x,t)=E(x;p)\cdot H(t), \quad \ldots\ldots \quad (392)$$

and accordingly obtain for $V(x;p)$ the ordinary differential equation

$$\frac{d}{dx}\left(\alpha(x)\cdot\frac{dV}{dx}\right)+\beta(x)\cdot V-p^2\cdot a(x)\cdot V=\gamma(x;p), \quad \ldots\ldots \quad (393)$$

where

$$\gamma(x;p)=E(x;p)-p^2\cdot a(x)\cdot F(x)-p\cdot a(x)\cdot G(x). \quad \ldots \quad (394)$$

Further, let $G(x,\xi)$ be the Green function of the differential form

$$\frac{d}{dx}\left(\alpha(x)\cdot\frac{dV}{dx}\right)+\beta(x)\cdot V \ldots\ldots\ldots\ldots\ldots \quad (395)$$

under the boundary conditions in question.

Performing the same reasoning as above when studying the equation of telegraphy, we easily prove the following integral equation to be satisfied by the function $V(x;p)$:

$$V(x)+p^2\cdot\int_0^s G(x,\xi)\cdot a(\xi)\cdot V(\xi)\cdot d\xi=Q(x;p), \quad \ldots\ldots \quad (396)$$

where we have put for brevity

$$Q(x;p)=-\int_0^s G(x,\xi)\cdot\gamma(\xi;p)\cdot d\xi. \quad \ldots\ldots\ldots\ldots \quad (397)$$

Unlike (353) this equation has a kernel $G(x,\xi)\cdot a(\xi)$ that is not symmetric, unless the density $a(\xi)$ is a constant. However, one can readily pass over to an integral equation with a symmetric kernel also in this case. For that scope we only have to multiply (396) by $\sqrt{a(x)}$ ($a(x)$ is supposed not to assume negative values in the interval $(0, s)$) and then to introduce the new functions:

$$\left.\begin{aligned} V_*(x) &= \sqrt{a(x)} \cdot V(x), \\ G_*(x, \xi) &= \sqrt{a(x)} \cdot \sqrt{a(\xi)} \cdot G(x, \xi), \\ Q_*(x; p) &= \sqrt{a(x)} \cdot Q(x; p). \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (398)$$

Then our equation will appear in the form

$$V_*(x) + p^2 \cdot \int_0^s G_*(x, \xi) \cdot V_*(\xi) \cdot d\xi = Q_*(x; p). \quad \ldots\ldots \quad (399)$$

The kernel now being real and symmetric, because the Green function $G(x, \xi)$ has these properties, we know that the corresponding homogeneous integral equation

$$\varphi(x) = \lambda \cdot \int_0^s G_*(x, \xi) \cdot \varphi(\xi) \cdot d\xi \quad \ldots\ldots\ldots\ldots \quad (400)$$

has an infinite set of real characteristic numbers. These may be proved to be all simple and positive; we will denote them by ω_1^2, $\omega_2^2 \ldots\ldots$ and the corresponding normal functions by $\varphi_1(x)$, $\varphi_2(x) \ldots\ldots$.

According to Schmidt's formula the solution of the inhomogeneous equation (396) can be written:

$$V_*(x) = Q_*(x; p) - \sum_{n=1}^{\infty} \frac{p^2}{p^2 + \omega_n^2} \cdot Q_n(p) \cdot \varphi_n(x), \quad \ldots\ldots \quad (401)$$

where

$$Q_n(p) = \int_0^s Q_*(\xi; p) \cdot \varphi_n(\xi) \cdot d\xi. \quad \ldots\ldots\ldots\ldots \quad (402)$$

Now we make both the members of (401) operate upon the unit-function $H(t)$. From (394), (397), and (398) follows:

$$
\begin{aligned}
Q_*(x;p)=\sqrt{a(x)}\cdot\int_0^s G(x,\xi)\cdot[p^2\cdot a(\xi)\cdot F(\xi)+ \\
+p\cdot a(\xi)\cdot G(\xi)-E(\xi;p)]\cdot d\xi,
\end{aligned}
\qquad (403)
$$

and we get, because $p=\dfrac{d}{dt}$ if $t>0$,

$$Q_*(x;p)\cdot H(t)=-\sqrt{a(x)}\cdot\int_0^s G(x,\xi)\cdot e(\xi,t)\cdot d\xi \;\ldots\ldots\; (404)$$

Further, by (402) we have

$$
\begin{aligned}
Q_n(p)=\int_0^s\Big\{\sqrt{a(\xi)}\cdot\int_0^s G(\xi,\eta)\cdot[p^2\cdot a(\eta)\cdot F(\eta)+ \\
+p\cdot a(\eta)\cdot G(\eta)-E(\eta;p)]\cdot d\eta\Big\}\cdot\varphi_n(\xi)\cdot d\xi.
\end{aligned}
\qquad (405)
$$

Hence we may write:

$$Q_n(p)=p^2\cdot F_n+p\cdot G_n-E_n(p) \;\ldots\ldots\ldots\ldots\; (406)$$

Employing the notations

$$F_*(x)=\sqrt{a(x)}\cdot F(x),\quad G_*(x)=\sqrt{a(x)}\cdot G(x), \;\ldots\ldots\; (407)$$

we can define the coefficients F_n and G_n by the double integrals:

$$
\left.
\begin{aligned}
F_n=\int_0^s\int_0^s G_*(\xi,\eta)\cdot F_*(\eta)\cdot\varphi_n(\xi)\cdot d\xi\cdot d\eta, \\
G_n=\int_0^s\int_0^s G_*(\xi,\eta)\cdot G_*(\eta)\cdot\varphi_n(\xi)\cdot d\xi\cdot d\eta.
\end{aligned}
\right\}\;\ldots\; (408)
$$

On account of the symmetry relation $G_*(\xi,\eta)=G_*(\eta,\xi)$ and because $\varphi_n(\xi)$ satisfies the homogeneous equation, we can perform one integration, thus obtaining the simpler expressions:

$$\left.\begin{aligned} F_n&=\frac{1}{\omega_n^2}\cdot\int_0^s F_*(\eta)\cdot\varphi_n(\eta)\cdot d\eta\equiv\frac{1}{\omega_n^2}\cdot F_n^*,\\ G_n&=\frac{1}{\omega_n^2}\cdot\int_0^s G_*(\eta)\cdot\varphi_n(\eta)\cdot d\eta\equiv\frac{1}{\omega_n^2}\cdot G_n^*.\end{aligned}\right\}\ \ldots\ldots \tag{409}$$

The last term of (406) is defined by the integral

$$E_n(p)=\int_0^s\int_0^s G(\xi,\eta)\cdot\sqrt{a(\xi)}\cdot E(\eta;p)\cdot\varphi_n(\xi)\cdot d\xi\cdot d\eta. \ \ldots \tag{410}$$

With our new notations the formula (401) presents us the function $v(x,t)$ in the form

$$\left.\begin{aligned} v(x,t)&=-\sqrt{a(x)}\cdot\int_0^s G(x,\xi)\cdot e(\xi,t)\cdot d\xi\\ &-\sum_{n=1}^{\infty}(p^2\cdot F_n+p\cdot G_n)\cdot\frac{p^2}{p^2+\omega_n^2}\cdot H(t)\cdot\varphi_n(x)\\ &+\sum_{n=1}^{\infty}\frac{p^2}{p^2+\omega_n^2}\cdot E_n(p)\cdot H(t)\cdot\varphi_n(x).\end{aligned}\right\}\ \ldots \tag{411}$$

At last, making use of rule nr 5 of our table and the notations (409), we arrive at the explicit function $v(x,t)$:

$$v(x,t) = -\sqrt{a(x)}\cdot\int_0^s G(x,\xi)\cdot e(\xi,t)\cdot d\xi$$

$$+\sum_{n=1}^{\infty} F_n^*\cdot\varphi_n(x)\cdot\cos\omega_n t\cdot H(t) + \sum_{n=1}^{\infty}\frac{G_n^*}{\omega_n}\cdot\varphi_n(x)\cdot\sin\omega_n t\cdot H(t) \qquad (412)$$

$$+\sum_{n=1}^{\infty}\varphi_n(x)\cdot\left[e_n(t) - \omega_n\cdot\int_0^t \sin\omega_n(t-\tau)\cdot e_n(\tau)\cdot d\tau\right].$$

Here the first term may be developed into a series of normal functions, thus being found to cancel the first part of the third sum in (412). Therefore, on introducing

$$e_n(t) = \frac{1}{\omega_n^2}\cdot\int_0^s \frac{e(\eta,t)}{\sqrt{a(\eta)}}\cdot\varphi_n(\eta)\cdot d\eta = \frac{1}{\omega_n^2}\cdot e_n^*(t), \ldots\ldots\ldots \qquad (413)$$

which expression is easily obtained by applying $E_n(p)$ to $H(t)$, we get the final result:

$$v(x,t) = -\sum_{n=1}^{\infty}\frac{1}{\omega_n}\cdot\varphi_n(x)\cdot\int_0^t \sin\omega_n(t-\tau)\cdot e_n^*(\tau)\,d\tau$$
$$+\sum_{n=1}^{\infty} F_n^*\cdot\varphi_n(x)\cdot\cos\omega_n t\cdot H(t) + \sum_{n=1}^{\infty}\frac{G_n^*}{\omega_n}\cdot\varphi_n(x)\cdot\sin\omega_n t\cdot H(t). \qquad (414)$$

This involves, as we see, the Fourier-coefficients of the functions

$$\sqrt{a(x)}\cdot F(x),\ \sqrt{a(x)}\cdot G(x),\ \text{and}\ \frac{e(x,t)}{\sqrt{a(x)}},$$

which, together with the boundary conditions (391), specify the problem.

Appendix.

1. The Fourier-Mellin integral theorem.

Assume the real function $f(t)$ of the real variable t to have the following properties:

1. $f(t)$ and its first derivative are piecewise continuous in each finite interval.

2. $f(t)=\frac{1}{2}[f(t+0)+f(t-0)]$ at a point of discontinuity.

3. The integral $\int\limits_{-\infty}^{+\infty}|f(\tau)|\cdot d\tau$ exists.

Then the classical Fourier theorem states that the function $f(t)$ can be identically represented by the double integral

$$f(t)=\frac{1}{\pi}\int\limits_{0}^{\infty}d\lambda\int\limits_{-\infty}^{+\infty}f(\tau)\cdot\cos\lambda(\tau-t)\cdot d\tau, \qquad (1)$$

which may also be written

$$f(t)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}d\lambda\int\limits_{-\infty}^{+\infty}f(\tau)\cdot e^{-i\lambda(\tau-t)}\cdot d\tau, \qquad (1\,a)$$

provided that the right hand side of (1) converges if the cos-function is replaced by the sin-function; its value then equals zero, because sin is an odd function.

By introducing the new variable $p=i\lambda$, we can also express the contents of (1 a) by the two formulas

$$f(t)=\frac{1}{2\pi i}\int\limits_{-i\infty}^{+i\infty}F(p)\cdot\frac{e^{pt}}{p}\cdot dp, \qquad (2)$$

$$F(p)=p\cdot\int_{-\infty}^{+\infty} f(\tau)\cdot e^{-p\tau}\cdot d\tau, \quad \ldots\ldots\ldots\ldots \quad (2\,a)$$

the path of integration in the first integral being the imaginary axis of the complex p-plane.

As to the proof of Fourier's integral theorem we refer to textbooks of the subject, e. g. Courant-Hilbert, Methoden der matematischen Physik.

The Mellin modification of this theorem, as formulated in Chapter 1, formulas (1) and (2), is now readily obtained in the following way.

In the right hand member of the first of these formulas, which by the second may be written

$$R.\,M.=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{pt}\cdot dp\int_{-\infty}^{+\infty} f(\tau)\, e^{-p\tau}\cdot d\tau,$$

we put $p=\sigma+i\lambda$, σ being a real constant, λ a real variable. Hence $dp=i\cdot d\lambda$, and our path of integration runs parallel to the imaginary axis. Thus we obtain

$$R.\,M.=e^{\sigma t}\cdot\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\lambda\int_{-\infty}^{+\infty} f(\tau)\cdot e^{-\sigma\tau}\cdot e^{-i\lambda(\tau-t)}\cdot d\tau$$

$=e^{\sigma t}\cdot f(t)\cdot e^{-\sigma t}=f(t)$, according to Fourier's theorem (1 a), provided that the function $f(t)\cdot e^{-\sigma t}$ fulfils the condition 3 of integrability, i. e. that $\int_{-\infty}^{+\infty}|f(\tau)|\cdot e^{-\sigma\tau}\cdot d\tau$ exists. When this is the case if $\alpha_1<\sigma<\alpha_2$, any line parallel to the imaginary axis in the band region $\alpha_1<Re\{p\}<\alpha_2$ may be used as path of integration.

A theorem, reciprocal to that just proved, is also due to Mellin. This may be stated as follows:

Let $F(p)$ be a function of the complex variable p such that: $(p=\sigma+i\lambda)$

1. $\frac{F(p)}{p}$ is holomorphical in the band region $\alpha<\sigma<\beta$.

2. $\int\limits_{-\infty}^{+\infty} \left| \frac{F(p)}{p} \right| d\lambda$ converges in the same region.

3. $\left| \frac{F(p)}{p} \right| \to 0$ uniformly, when $\lambda \to \pm \infty$ in the region

$$\alpha+\delta \leq \sigma \leq \beta-\delta, \ (\delta>0).$$

Then formula 2 of Chapter 1 is a consequence of formula 1. We refer to the book mentioned above for a more complete examination of these so-called *reciprocal theorems.*

As a rule our investigations are to be performed on functions $f(t)$ that have no other point of discontinuity than possibly the origin; for $t<0$ they will vanish identically and for $t>0$ be more or less regular. Naturally, the integral theorem ought to be proved comparatively simply for such functions. We will now derive the theorem in a form suitable in many cases, and in a way that will give us a wider view of the notions of basic integrator and integral operator of Chapters 2 and 3.

We start from the expression

$$\psi(t)=f(0)\cdot\varphi(t)+\int\limits_0^t f'(t-\tau)\cdot\varphi(\tau)\cdot d\tau, \ \dots\dots\dots\dots \quad (3)$$

$\varphi(t)$ being any integrable function and $f(t)$ a function having continuous derivatives in the interval of variability employed. We are going to perform a suite of integrations by parts, and for that aim we introduce the auxiliary notations

$$u_1(t)=\int\limits_0^t \varphi(\tau)\cdot d\tau, \dots \quad u_n(t)=\int\limits_0^t u_{n-1}(\tau)\cdot d\tau. \ \dots\dots \quad (4)$$

It is easily proved by the Lejeune-Dirichlet's method that

$$u_n(t)=\frac{1}{\underline{|n-1}}\cdot\int\limits_0^t (t-\tau)^{n-1}\cdot\varphi(\tau)\cdot d\tau. \dots\dots\dots\dots \quad (5)$$

As all the $u_n(0)$ vanish, the repeated integration by parts gives

$$\psi(t)=$$

$$=f(0)\cdot\varphi(t)+f'(0)\cdot u_1(t)+\ldots.+f^{(n-1)}(0)\cdot u_{n-1}(t)+$$

$$+\int_0^t f^{(n)}(t-\tau)\cdot u_{n-1}(\tau)\cdot d\tau,$$

and therefore we get, according to (5),

$$\begin{aligned}\psi(t)&=f(0)\cdot\varphi(t)\\&+\int_0^t\left\{f'(0)+\frac{f''(0)}{\underline{|1}}\cdot(t-\tau)+\cdots+\frac{f^{(n-1)}(0)}{\underline{|n-2}}\cdot(t-\tau)^{n-2}\right\}\cdot\varphi(\tau)\cdot d\tau\\&+\frac{1}{\underline{|n-2}}\int_0^t f^{(n)}(t-\tau)\cdot\left\{\int_0^\tau(\tau-\xi)^{n-2}\cdot\varphi(\xi)\cdot d\xi\right\}\cdot d\tau.\end{aligned}\tag{6}$$

The last term may be transformed in the above manner, thus getting the form:

$$\frac{1}{\underline{|n-2}}\int_0^t\left\{\int_\tau^t f^{(n)}(t-\xi)\cdot(\xi-\tau)^{n-2}\cdot d\xi\right\}\cdot\varphi(\tau)\cdot d\tau.\ \ldots\ldots\tag{7}$$

Obviously, the function $\psi(t)$ can be regarded as derived from $\varphi(t)$ by a symbolic multiplication (Cf. Chapter 3):

$$\psi(t)=F\times\varphi(t),\ldots\ldots\ldots\ldots\ldots\ldots\ldots\tag{8}$$

the integral operator F being

$$\begin{aligned}F=\Bigg[f(0)+\int_0^t\Bigg\{&f'(0)+f''(0)\cdot\frac{t-\tau}{\underline{|1}}+\ldots.+f^{(n-1)}(0)\cdot\frac{(t-\tau)^{n-2}}{\underline{|n-2}}+\\&+\frac{1}{\underline{|n-2}}\int_\tau^t f^{(n)}(t-\xi)\cdot(\xi-\tau)^{n-2}\cdot d\xi\Bigg\}\cdot d\tau\Bigg].\end{aligned}\tag{9}$$

The case $n=1$ corresponds to the original formula (3), which gives

$$F=\left[f(0)+\int\limits_0^t f'(t-\tau)\cdot d\tau\right],$$

i. e. the formula (36) of Chapter 3.

Now we return to formula (6), specializing $\varphi(t)$ to the unit-function $H(t)$. Then we know *à priori* that $\psi(t)=f(t)$, provided that $f(t)=0$ when $t<0$. All integrations but one may now be performed and we get the simpler formula

$$\begin{aligned} f(t)=&\left\{f(0)+f'(0)\cdot\frac{t}{\underline{|1}}+\cdots+f^{(n-1)}(0)\cdot\frac{t^{n-1}}{\underline{|n-1}}\right\}\cdot H(t) \\ &+\frac{1}{\underline{|n-1}}\cdot\int\limits_0^t f^{(n)}(t-\tau)\cdot\tau^{n-1}\cdot H(\tau)\cdot d\tau. \end{aligned} \tag{10}$$

Now, it is possible to prove by means of the theory of residues and Jordan's lemma that, if m is a positive integer or zero and $\sigma>0$,

$$\frac{t^m}{\underline{|m}}\cdot H(t)=\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{1}{p^m}\cdot\frac{e^{pt}}{p}\cdot dp. \ldots\ldots\ldots\ldots\ldots\ldots \tag{11}$$

Accordingly we obtain

$$f(t)=\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\left\{f(0)+\frac{f'(0)}{p}+\cdots+\frac{f^{(n-1)}(0)}{p^{n-1}}\right\}\cdot\frac{e^{pt}}{p}\cdot dp+R_n.\,.\,. \tag{12}$$

The rest-term may be transformed in the following way:

$$\begin{aligned} R_n=&\frac{1}{\underline{|n-1}}\int\limits_0^t f^{(n)}(\tau)\cdot(t-\tau)^{n-1}\cdot H(t-\tau)\cdot d\tau \\ =&\frac{1}{\underline{|n-1}}\int\limits_0^\infty f^{(n)}(\tau)\cdot(t-\tau)^{n-1}\cdot H(t-\tau)\cdot d\tau= \end{aligned} \tag{13}$$

$$
\begin{aligned}
&= \int_0^{\infty} f^{(n)}(\tau) \cdot \left\{ \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{p(t-\tau)}}{p^n} \cdot dp \right\} d\tau \\
&= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \left\{ \int_0^{\infty} f^{(n)}(\tau) \cdot e^{-p\tau} \cdot d\tau \right\} \cdot \frac{e^{pt}}{p^n} \cdot dp.
\end{aligned}
\qquad (13)
$$

The last operation, viz the inversion of the order of integration, is correct, if the double integral converges absolutely.* This will certainly be the case, if $f^{(n)}(t)$ belongs to class nr n, and if $\sigma > \alpha$ (Cf. the definition in Chapter 1; $|f^{(n)}(t)| < e^{\alpha t}$ when $t > t_*$). Indeed, putting $p = \sigma + i \cdot \omega$, we get as an upper bound of the magnitude of the last expression:

$$
e^{\sigma t} \cdot \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ \int_0^{\infty} |f^{(n)}(\tau)| \cdot e^{-\sigma\tau} \cdot d\tau \right\} \frac{d\omega}{(\sigma^2 + \omega^2)^{\frac{n}{2}}}.
$$

Even the outer integral must converge, if $n > 1$.

If (13) be introduced into (12), we get the result:

$$
\left.
\begin{aligned}
f(t) &= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(p) \cdot \frac{e^{pt}}{p} \cdot dp, \\
F(p) &= f(0) + \frac{f'(0)}{p} + \cdots + \frac{f^{(n-1)}(0)}{p^{n-1}} + \frac{1}{p^{n-1}} \cdot \int_0^{\infty} f^{(n)}(\tau) \cdot e^{-p\tau} \cdot d\tau.
\end{aligned}
\right| \qquad (14)
$$

This equality constitutes the integral theorem in the wanted form. By our deductions we have got it proved for any function of class 2, but as we know, it also subsists for functions of class 1. The last equality (14) is, of course, nothing but formula (20) in Chapter 1.

By a quite analogous proceeding we may generally prove the identity (Cf. Chapter 3)

$$
F \times \varphi(t) = F(p) \cdot \varphi(t) \quad \ldots\ldots\ldots\ldots\ldots \quad (15)
$$

---

* This condition is sufficient but not necessary.

to be valid for any function $\varphi(t)$, the basic integrator of which (Cf. Chapter 2) may be $\Phi(p)$. In fact, we only need to employ the rule (Cf. page 36)

$$\frac{1}{\underline{|n-1}}\int_{0}^{t}(t-\tau)^{n-1}\cdot\varphi(\tau)\cdot d\tau=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Phi(p)\cdot\frac{1}{p^{n}}\cdot\frac{e^{pt}}{p}\cdot dp,\;\ldots\ldots\quad(16)$$

which is a direct consequence of the rule of integration on page 15.

The expression $\psi(t)=F\times\varphi(t)$ has the meaning defined by (6). From (16) follows that

$$\psi(t)=\qquad(17)$$
$$=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{f(0)+\frac{f'(0)}{p}+\frac{f''(0)}{p^{2}}+\ldots\ldots+\frac{f^{(n-1)}(0)}{p^{n-1}}\right\}\cdot\Phi(p)\cdot\frac{e^{pt}}{p}\cdot dp+R_{n},$$

where, in accordance with (16), we may write

$$R_{n}=\int_{0}^{t}f^{(n)}(t-\tau)\cdot\left\{\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Phi(p)\cdot\frac{e^{p\tau}}{p^{n}}\cdot dp\right\}\cdot d\tau.\;\ldots\ldots\quad(18)$$

Transforming as above, we get

$$R_{n}=\int_{0}^{\infty}f^{(n)}(\tau)\cdot\left\{\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Phi(p)\cdot\frac{e^{p(t-\tau)}}{p^{n}}\cdot dp\right\}\cdot d\tau$$
$$\qquad(19)$$
$$=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{\int_{0}^{\infty}f^{(n)}(\tau)\cdot e^{-p\tau}\cdot d\tau\right\}\cdot\Phi(p)\cdot\frac{e^{pt}}{p^{n}}\cdot dp.$$

Because $|\Phi(p)|<A$ when $\omega>\Omega$, the absolute convergence of the double integral can be proved just as above, if $n>1$. Adding this result to (17), we get, according to (14),

$$\psi(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}F(p)\cdot\Phi(p)\cdot\frac{e^{pt}}{p}\cdot dp.\;\ldots\ldots\ldots\quad(20)$$

But here, by definition, the right member agrees with the function $F(p) \cdot \varphi(t) = F(p) \cdot \Phi(p) \cdot H(t)$.

The expressions (9) of F and (14) of $F(p)$ may conveniently be termed the *transformed* operator resp. integrator *of order n.*

2. A proof of Jordan's lemma.

Let $\Phi(p)$ be a function that fulfils the conditions stated on page 10 in the region $Re\{p\} \geq \alpha$. The path of integration may be a half-circle C of radius ϱ and with its centre at m, where $Re\{m\} \geq \alpha$. Then upon C we can put

$$p = m + \varrho \cdot e^{i\theta}.$$

Denoting by ε an arbitrarily small positive number, we can always find, on account of our assumptions, a number $\varrho_0(\varepsilon)$ such that the inequality

$$|\Phi(p)| < \frac{\varepsilon}{\pi}$$

holds at all points upon C, as far as $\varrho \geq \varrho_0$. Then we get the following estimation:

$$\left| \int_C \Phi(p) \cdot e^{pt} \cdot dp \right| < \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\varepsilon}{\pi} \cdot \left| e^{(m + \varrho \cos\theta + i \varrho \sin\theta) \cdot t} \right| \cdot \varrho \cdot d\theta$$

or:

$$\left| \int_C \Phi(p) \cdot e^{pt} \cdot dp \right| < \frac{2\varepsilon \cdot \varrho}{\pi} \cdot e^{Re\{m\} \cdot t} \cdot \int_{-\frac{\pi}{2}}^{0} e^{\varrho \cdot \cos\theta \cdot t} \cdot d\theta.$$

Now we introduce $\theta = \theta' - \frac{\pi}{2}$ and make use of the fact that $\sin\theta' \geq \frac{2}{\pi} \cdot \theta'$ in the interval $0 \leq \theta' \leq \frac{\pi}{2}$. Hence we obtain, if $t < 0$:

$$\left| \int_C \Phi(p) e^{pt} dp \right| < \frac{2\varepsilon\varrho}{\pi} \cdot e^{Re\{m\} \cdot t} \cdot \int_0^{\frac{\pi}{2}} e^{\frac{2t\varrho}{\pi} \cdot \theta'} \cdot d\theta' =$$

$$= \frac{2}{\pi} \cdot \varepsilon \cdot \varrho \cdot e^{Re\{m\} \cdot t} \cdot \frac{\pi}{2t\varrho} \cdot \left[e^{\frac{2\varrho t}{\pi} \cdot \theta'} \right]_0^{\frac{\pi}{2}},$$

or

$$\left|\int_C \Phi(p)\cdot e^{pt}\cdot dp\right| < -\frac{\varepsilon}{t}\cdot e^{Re\{m\}\cdot t} \quad . \ \dots\dots\dots\dots \quad (1)$$

Also this inequality subsists for all $\varrho > \varrho_0(\varepsilon)$. However small the value of $|t| \neq 0$ is, we can always choose an ε so that the right hand side is $< \varepsilon_1$, ε_1 being an arbitrarily small positive number. In fact, if

$$\varepsilon < \varepsilon_1\cdot|t|\cdot e^{Re\{m\}\cdot|t|}, \quad \dots\dots\dots\dots\dots\dots \quad (2)$$

the absolute value of the integral must be less than ε_1, and this for all half-circles with $\varrho > \varrho_0(\varepsilon_1, t)$. Obviously this result involves the statement of Jordan's lemma. From (1) follows that the convergence is uniform in the interval $t \leq t_0 < 0$, provided that $Re\{m\} > 0$.

3. A proof of the theorems of page 8.

These theorems refer to a function of the complex variable p, presented in the form of a definite integral, viz

$$\frac{F(p)}{p} = \int_0^\infty f(\tau)\cdot e^{-p\tau}\cdot d\tau, \quad \dots\dots\dots\dots\dots \quad (1)$$

$f(\tau)$ being a real function of class nr 1 at least (Cf. p. 7).

The first theorem states that $\frac{F(p)}{p}$ is holomorphical in the region $Re\{p\} > \alpha$.

To prove this we will first verify that the infinite integrals

$$I_1 = \int_0^\infty f(\tau)\cdot e^{-p\tau}\cdot d\tau \quad \text{and} \quad I_2 = \int_0^\infty \tau\cdot f(\tau)\cdot e^{-p\tau}\cdot d\tau \ \dots\dots \quad (2)$$

are uniformly convergent if $Re\{p\} \geq \sigma_0 > \alpha$.

This may be done by estimating the rest-integrals I_1' and I_2' with the limits t and ∞. We obtain:

$$|I_1'| = \left|\int_t^\infty f(\tau)\cdot e^{-p\tau}\cdot d\tau\right| < \int_t^\infty e^{-(\sigma_0-\alpha)\cdot\tau}\cdot d\tau = \frac{1}{\sigma_0-\alpha}\cdot e^{-(\sigma_0-\alpha)\cdot t}, \quad (3)$$

provided that $t > \tau_*$ (Cf. (3) in Chapter 1). Here the right hand side can be made less than any positive number ε if t is made sufficiently large, say if $t \geq t_1 > \tau_*$, t_1 being the same for any p of the region $Re\{p\} \geq \sigma_0 > \alpha$.

Further we have, if $t > \tau_*$,

$$|I_2'| = \left| \int_t^\infty \tau \cdot f(\tau) \cdot e^{-p\tau} \cdot d\tau \right| < \int_t^\infty \tau \cdot e^{-(\sigma_0-\alpha)\cdot\tau} \cdot d\tau =$$

$$= \frac{(\sigma_0-\alpha)\cdot t+1}{(\sigma_0-\alpha)^2} \cdot e^{-(\sigma_0-\alpha)\cdot t}. \qquad \ldots\ldots\ldots\ldots\ldots \qquad (4)$$

Obviously, as $x \cdot e^{-x}$ tends to zero if x tends to infinity, the right hand side of (4) will remain less than ε if $t \geq t_2 > \tau_*$, t_2 being the same number all over the region $Re\{p\} \geq \sigma_0 > \alpha$.

Now, as I_2 converges uniformly and its integrand $\tau \cdot f(\tau) \cdot e^{-p\tau}$ is a continuous function of τ and p, we may conclude that I_2, but for the sign, represents the derivative of I_1 with respect to p. Hence the function $\frac{F(p)}{p}$ has a derivative at each point, and accordingly it must be analytic anywhere in the region $Re\{p\} > \alpha$.

The second theorem states that $\frac{F(p)}{p}$ tends uniformly to zero if $|p| \to \infty$ in the half-plane $Re\{p\} \geq \sigma_0 > \alpha$. For the proof we will employ the formula (14) in Chapter 2:

$$F(p) = f(0) + \int_0^\infty f'(\tau) \cdot e^{-p\tau} \cdot d\tau.$$

Of course, we have only to verify that the last term of this expression is uniformly limited within the region in question. We obtain:

$$\left| \int_0^\infty f'(\tau) \cdot e^{-p\tau} \cdot d\tau \right| \leq \left| \int_0^{\tau_*} f'(\tau) \cdot e^{-p\tau} \cdot d\tau \right| + \left| \int_{\tau_*}^\infty f'(\tau) \cdot e^{-p\tau} \cdot d\tau \right|$$

$$< M\cdot\int_{0}^{\tau_*} e^{-\sigma_0\tau}\cdot d\tau + \int_{\tau_*}^{\infty} e^{-(\sigma_0-\alpha)\cdot\tau}\cdot d\tau,$$

denoting by M an upper bound of $|f'(\tau)|$ in the interval $(0-\tau_*)$. Accordingly

$$\left|\int_{0}^{\infty} f'(\tau)\cdot e^{-p\tau}\cdot d\tau\right| < \frac{M}{\sigma_0}\cdot(1-e^{-\sigma_0\cdot\tau_*})+\frac{1}{\sigma_0-\alpha}\cdot e^{-(\sigma_0-\alpha)\cdot\tau_*},\ \dots\dots \quad (5)$$

and the right hand side represents an upper bound of the magnitude of the integral, which is the same for all points of the region $Re\{p\} \geq \sigma_0 > \alpha$.

4. Note about the convergence of the modified integral.

It is almost trivial to prove the uniform convergence for $t \geq t_0$ (t_0 an arbitrarily small positive number) of the complex integral

$$I=\frac{1}{2\pi i}\int_{L_1L_2} \Phi(p)\cdot e^{pt}\cdot dp,\ \dots\dots\dots\dots\dots\dots \quad (1)$$

$\Phi(p)$ being a function that increases at most as a power of p, when $p \to \infty$ along L_1 and L_2 (Cf. page 17 and fig. 2).

Let us split the path of integration into three parts L'_1, $L''_1+L''_2$, L'_2 by the points P_1 and P_2, the first and the last part having an infinite extension. To get the uniform convergence established we only have to verify that it is always possible to fix the positions of P_1 and P_2 so that the contributions to the integral from L'_1 and L'_2 remain less in magnitude than an arbitrarily small positive number for all values of $t \geq t_0 > 0$. Obviously, it is sufficient to perform the proof for one of the parts, say L'_2.

Upon L_2 we may put $p=\sigma+\varrho\cdot e^{i\varphi}$ $(\sigma \geq \alpha)$. Accordingly $dp= d\varrho\cdot e^{i\varphi}$ and

$$e^{pt}=e^{(\sigma+\varrho\cdot\cos\varphi+i\varrho\cdot\sin\varphi)\cdot t}.$$

Therefore we get

$$|I'_2| \leq \frac{e^{\sigma t}}{2\pi}\cdot\int_{\varrho_2}^{\infty} |\Phi(p)|\cdot e^{\varrho\cdot\cos\varphi\cdot t}\cdot d\varrho,\ \dots\dots\dots\dots \quad (2)$$

denoting by I'_2 the contribution to I from L'_2.

Because of our condition that $|\Phi(p)|$ should increase at most as rapidly as a power of $|p|$ when $\varrho \to \infty$, there must surely exist a number ϱ_2^* such that the inequality

$$|\Phi(p)| < e^{k\varrho} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

subsists for all values of $\varrho \geq \varrho_2^*$, k being any fixed positive number. Hence we obtain, taking $\varrho_2 \geq \varrho_2^*$,

$$|I'_2| < \frac{e^{\sigma t}}{2\pi} \int_{\varrho_2}^{\infty} e^{(k+t\cdot\cos\varphi)\cdot\varrho} \cdot d\varrho. \quad \ldots\ldots\ldots\ldots \quad (4)$$

This integral converges if $k+t\cdot\cos\varphi < 0$, i. e. if

$$t\cdot(-\cos\varphi) > k,$$

and we get

$$|I'_2| < \frac{1}{2\pi} \cdot \frac{e^{k\varrho_2+(\sigma+\varrho_2\cdot\cos\varphi)\cdot t}}{t\cdot(-\cos\varphi)-k} \leq \frac{1}{2\pi} \cdot \frac{e^{k\varrho_2+(\sigma+\varrho_2\cdot\cos\varphi)\cdot t_0}}{t_0\cdot(-\cos\varphi)-k}, \quad \ldots\ldots \quad (5)$$

provided that

$$t \geq t_0 > \frac{k}{-\cos\varphi} \text{ and } \varrho_2 > \frac{\sigma}{-\cos\varphi}. \quad \ldots\ldots\ldots\ldots \quad (6)$$

Accordingly, we have got $|I'_2| <$ an expression that may be made arbitrarily small if we take ϱ_2 sufficiently great, and this for all values of $t \geq t_0$. It should be observed that the bound t_0 can be regarded as arbitrarily small, since k could be any positive number.

The integral I'_1 can be treated in exactly the same way, and hence the uniform convergence of I is proved.

In the modified integral (β) on page 17 we have $\Phi(p) = Q(p)\cdot\frac{F(p)}{p}$, where $Q(p)$ is assumed to increase at most as a power of p and $\frac{F(p)}{p}$ to tend to zero when $p \to \infty$ in the region in question. Thus the uniform convergence of the integral (β) follows *à fortiori* from that of I.

5. On the examination of more general irrational integrators. Generalization of the expansion theorems.

In Chapter 9 we investigated a class of irrational integrators possessing a single branch-point at the origin. The results thus gained may be generalized without difficulty to integrators $F(p)$ with an arbitrary but finite number of poles and branch-points.

To arrive at an expansion theorem corresponding to the formulas (130) and (134) of Chapter 7 we will assume $F(p)$ to be a basic integrator with the following properties:

$F(p)$ shall satisfy the condition A of Chapter 2. Upon the path of integration L_1 L_2 (Cf. fig. 11), into which the original path L can be changed if $t > 0$, and to the left of which all the singularities of $\frac{F(p)}{p}$ must be situated, the expansion

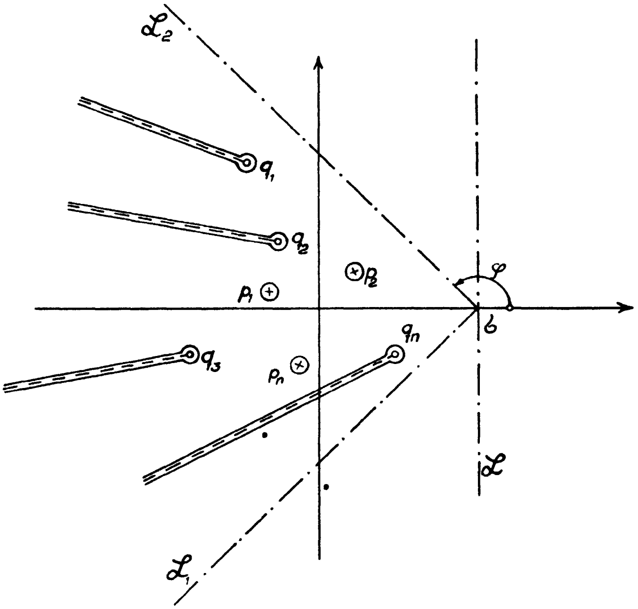

Fig. 11.

$$F(p) = \sum_{\nu=1}^{\infty} \frac{a_\nu}{p^{s_\nu}} \tag{1}$$

$$-1 < s_1 < s_2 < \cdots$$

shall be valid under uniform convergence.

In order that $F(p)$ shall be uniquely defined we assume cuts e. g. in the form of straight lines to be put between the branch-points $q_1, q_2 \cdots$ and the point at infinity in the domain to the left of L_1 and L_2; obviously these cuts can always be drawn so that they do not intersect.

Now we shall prove the expansion theorem:

$$f(t)=F(p)\cdot H(t)=\sum_{\nu=1}^{\infty}\frac{a_\nu}{p^{s_\nu}}\cdot H(t)=\sum_{\nu=1}^{\infty}a_\nu\cdot\frac{t^{s_\nu}}{\Pi(s_\nu)}\cdot H(t). \quad \ldots\ldots \quad (2)$$

At first we may note that, according to our assumptions, $f(t)=0$ when $t<0$. To evaluate the integral

$$f(t)=\frac{1}{2\pi i}\int_{L_1L_2}F(p)\cdot\frac{e^{pt}}{p}\cdot dp \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

when $t>0$ we introduce:

$$F(p)=\sum_{\nu=1}^{n}\frac{a_\nu}{p^{s_\nu}}+R_n(p), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

where, ε being an arbitrarily small positive number, we shall have

$$|R_n(p)|<\varepsilon$$

for all values of n sufficiently large, say $n>N(\varepsilon)$, N being the same for all points of L_1 and L_2.

Hence we get $(t>0)$:

$$f(t)=\sum_{\nu=1}^{n}a_\nu\cdot\frac{1}{2\pi i}\int_{L_1L_2}\frac{1}{p^{s_\nu}}\cdot\frac{e^{pt}}{p}\cdot dp+\frac{1}{2\pi i}\cdot\int_{L_1L_2}R_n(p)\cdot\frac{e^{pt}}{p}\cdot dp \quad \ldots\ldots \quad (5)$$

A rough estimation of the magnitude of the last term in (5), sufficiently good for our purpose, can easily be obtained, if we observe that on the path of integration we may put

$$p=\sigma+\varrho\cdot e^{\pm i\varphi},\quad |p|\geq|\sigma|\cdot\sin\varphi,\quad |e^{pt}|=e^{\sigma t}\cdot e^{\varrho\cdot\cos\varphi\cdot t},\quad |dp|=d\varrho$$

Thus:

$$\left|\frac{1}{2\pi i}\int_{L_1L_2}R_n(p)\cdot\frac{e^{pt}}{p}\cdot dp\right|<\frac{\varepsilon}{\pi\cdot\sin\varphi\cdot\cos\varphi}\cdot\frac{e^{\sigma t}}{t\cdot|\sigma|}. \quad \ldots\ldots\ldots \quad (6)$$

Accordingly, t being any fixed positive number, the right hand side of (6) can be made as small as we like by taking ε sufficiently small, i. e. by taking n sufficiently large. Therefore we have proved for $t > 0$ the equality:

$$f(t) = \sum_{\nu=1}^{\infty} a_\nu \cdot \frac{1}{2\pi i} \int_{L_1 L_2} \frac{1}{p^{s_\nu}} \cdot \frac{e^{pt}}{p} \cdot dp.$$

Here the origin is the common branch-point or pole of all the terms, and obviously we may replace the path $L_1 L_2$ by a loop enclosing the negative real axis. Thus (2) follows immediately.

We see at once that our previous rule

$$f(0) = F(\infty)$$

for functions $F(p)$ that are regular at infinity still subsists for the general class of basic integrators under consideration.

Now we shall proceed to investigate more generally the asymptotic form of a function $f(t) = F(p) \cdot H(t)$. For that aim we will first examine the contribution of a singularity at the origin by the same method as in Chapter 9, but this time assuming the cut to be any straight line from the origin to infinity in the left half-plane. (Cf. Fig. 12). Accordingly we take under consideration the integral

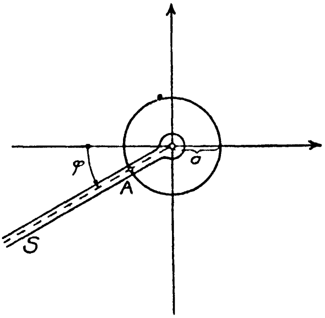

Fig. 12.

$$I = \frac{1}{2\pi i} \int_{S} F(p) \cdot \frac{e^{pt}}{p} \cdot dp, \quad \dots\dots\dots\dots\dots \quad (7)$$

extended over a loop S enclosing the cut. About the origin, say within the circle $|p| \leq a$, we prescribe $F(p)$ to be of the form

$$F(p) = \sum_{\nu=1}^{n} a_\nu \cdot p^{s_\nu} + p^s \cdot R_n, \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

where $s_1 < s_2 \cdots < s_n$, $s > 0$, and such that

$$|R_n| < \varepsilon \cdot \frac{\pi \cdot s}{a^s} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

for any sufficiently great value of n. Further we assume that it shall be possible to determine two real numbers $K > 0$ and k so that the inequality

$$|F(p)| < K \cdot e^{k \cdot |p|} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

subsists anywhere upon the borders of the cut outside the point A.

Our mode of reasoning will now be identically the same as in Chapter 9; we only have to change t into $t \cdot \cos \varphi$ in the formulas (191) and (199). Hence, by chosing n sufficiently great we must get

$$\left| I(t) - \sum_{\nu=1}^{n} a_\nu \cdot \frac{1}{2\pi i} \int_S p^{s_\nu} \cdot \frac{e^{pt}}{p} \cdot dp \right| < 3\varepsilon \quad \ldots\ldots\ldots\ldots \quad (11)$$

for all values of $t > t_1$ and t_2.

From the definition formula of a basic integrator we further conclude that

$$\begin{aligned} p^{s_\nu} \cdot H(t) &= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} p^{s_\nu} \cdot \frac{e^{pt}}{p} dp = \left[\frac{1}{2\pi i} \cdot \int_S p^{s_\nu} \cdot \frac{e^{pt}}{p} \cdot dp \right] \cdot H(t) = \\ &= \frac{t^{-s_\nu}}{\Pi(-s_\nu)} \cdot H(t) \end{aligned} \quad (12)$$

provided that $s_\nu < 1$. If $s_\nu \geq 1$ we may regard the integral along S as the modified integral, making it possible to maintain the last equality for all real values of s_ν, and thus it follows that (11) is equivalent to the asymptotic expansion (201) or (202) for $I(t)$.

The general case, where $\frac{F(p)}{p}$ has an arbitrary but finite number of singular points, can now be treated without difficulty. Obviously it means no essential limitation if, for the sake of simplicity, we confine ourselves to e. g. the case where $\frac{F(p)}{p}$ has the poles $p_0=0$, p_1, p_2 and the branch-points q_1, q_2. We also assume $\left|\frac{F(p)}{p}\right|$ to approach zero uniformly when $p\to\infty$, so that Jordan's lemma may be applied. Then we know that $f(t)\equiv 0$ when $t<0$, and

$$f(t)=I_{q_1}+I_{q_2}+I_{p_0}+I_{p_1}+I_{p_2}. \qquad\ldots\ldots\ldots\ldots (13)$$

when $t>0$, putting e. g.

$$I_{q_1}=\frac{1}{2\pi i}\int_{S_1} F(p)\cdot\frac{e^{pt}}{p}\cdot dp \qquad\ldots\ldots\ldots\ldots (14)$$

and

$$I_{p_1}=\frac{1}{2\pi i}\int_{\circ\, p_1} F(p)\cdot\frac{e^{pt}}{p}\cdot dp. \qquad\ldots\ldots\ldots\ldots (15)$$

Here S_1 denotes a loop surrounding a straight-line cut from the point q_1 to the point at infinity in the left half-plane (Cf. fig. 11).

To compute I_{q_1} we change the variable of integration, putting $p=z+q_1$, thus getting

$$I_{q_1}=e^{q_1\cdot t}\cdot\frac{1}{2\pi i}\int_{S_1-q_1} F(z+q_1)\cdot\frac{z}{z+q_1}\cdot\frac{e^{zt}}{z}\cdot dz. \qquad\ldots\ldots\ldots (16)$$

The path of integration in the z-plane becomes a loop surrounding the origin, obtained by a parallel displacement of S_1 a stretch given by the complex vector $-q_1$. Upon this path we may employ the results obtained above. For that aim we assume the equality:

$$F(z+q_1)\cdot\frac{z}{z+q_1}=\sum_{\nu=1}^{n} a_\nu\cdot z^{s_\nu}+z^{s}\cdot R_n \qquad\ldots\ldots\ldots\ldots (17)$$

to subsist within a circle of radius a about the origin, the factor R_n satisfying the condition (9). Of course, we may as well prescribe that

$$\frac{F(p)}{p}\cdot(p-q_1)=\sum_{\nu=1}^{n} a_\nu\cdot(p-q_1)^{s_\nu}+(p-q_1)^{s}\cdot R_n \quad \ldots\ldots \quad (17a)$$

in the region $|p-q_1| \leq a$ about q_1 in the p-plane. This being the case, we get from (16), as we know, the asymptotic expansion

$$I_{q_1} \backsim e^{q_1 t}\cdot\sum_{\nu=1}^{n} a_\nu\cdot\frac{t^{-s_\nu}}{\Pi(-s_\nu)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

or, symbolically written,

$$I_{q_1} \backsim e^{q_1 t}\cdot\sum_{\nu=1}^{n} a_\nu\cdot p^{s_\nu}\cdot H(t). \quad \ldots\ldots\ldots\ldots\ldots \quad (18a)$$

One can arrive at this expression in a simpler way by employing formally a shifting rule of our table in Chapter 5, viz:

$$F(p)\cdot H(t)=e^{\alpha t}\cdot F(p+\alpha)\cdot e^{-\alpha t}\cdot H(t).$$

Here we put $\alpha=q_1$ and $\frac{p}{p+q_1}\cdot H(t)$ instead of $e^{-q_1 t}\cdot H(t)$, thus getting

$$F(p)\cdot H(t)=e^{q_1 t}\cdot F(p+q_1)\cdot\frac{p}{p+q_1}\cdot H(t) \quad \ldots\ldots\ldots\ldots \quad (19)$$

After that we only have to employ the equality (17), replacing z by p, to recover (18a).

In a quite similar manner the constituent I_{q_2}, corresponding to the other branch-point q_2, is obtained.

The contributions I_{p_0}, I_{p_1}, I_{p_2} from the poles can, of course, be computed by the same method, but in this case *exact* instead of asymptotic expressions are obtained. It may be of some interest to compare this mode of proceeding with other methods of computing $I_{p_1}=(F(p)\cdot H(t))_{p_1}$.

We assume $F(p)$ to be of the form

$$\begin{aligned}(F(p))_{p_1}=b_2\cdot\frac{p}{(p-p_1)^2}+b_1\cdot\frac{p}{p-p_1}+a_0\cdot p+a_1\cdot p\,(p-p_1)+\\ +a_2\cdot p\,(p-p_1)^2+\cdots\end{aligned} \quad (20)$$

in the neighbourhood of the pole. Accordingly (Cf. (17a)),

$$\frac{F(p)}{p}\cdot(p-p_1)=b_2\cdot\frac{1}{p-p_1}+b_1+a_0\cdot(p-p_1)+a_1\cdot(p-p_1)^2+\cdots,$$

and we get by formula (18a)

$$I_{p_1}=e^{p_1 t}\cdot\left(b_2\cdot\frac{1}{p}+b_1+a_0\cdot p+a_1\cdot p^2+\cdots\right)\cdot H(t)=e^{p_1 t}\cdot(b_1+b_2\cdot t)\cdot H(t). \quad (21)$$

Alternatively we may apply the shifting rule to (20), thus getting more directly

$$(F(p)\cdot H(t))_{p_1}=e^{p_1 t}\cdot\left(b_2\cdot\frac{1}{p}+b_1+a_0\cdot p+a_1\cdot p^2+\cdots\right)\cdot H(t)=$$
$$=e^{p_1 t}\cdot(b_1+b_2\cdot t)\cdot H(t).$$

The simplest way of arriving at this result is, of course, to apply the rule nr 3 of the table to the principal part of (20) and to interpret p as $\frac{d}{dt}$ in the other terms.

Our investigations have brought us asymptotic or exact expressions of the different constituents of $f(t)$. As we see, every singular point of $\frac{F(p)}{p}$ leads to an exponential factor, and obviously the term corresponding to the singularity to the extreme right will dominate over the other terms more and more as t tends to infinity, thus representing the ultimate form of the function $f(t)$. If there are several such singularities, thus situated upon a line parallel to the imaginary axis, of course the sum of their contributions determines the final character of $f(t)$.

As an application of the general method developed we shall now derive the asymptotic expansion of the function

$$f(t)=F(p)\cdot H(t)=\frac{p}{\sqrt{p^2+1}}\cdot H(t). \quad \ldots\ldots\ldots\ldots \quad (22)$$

(Cf. pages 89—90). According to the uniformity-rule we have

$$f(\omega t)=F\left(\frac{p}{\omega}\right)\cdot H(t)=\frac{p}{\sqrt{p^2+\omega^2}}\cdot H(t)=J_0(\omega t)\cdot H(t).$$

Here we have two branch-points, $q_1=i$ and $q_2=-i$, but no pole. To get the contribution from q_1 we first transform (22) by the shifting-rule, thus obtaining (Cf. (19)):

$$f(t)=e^{it}\cdot\frac{\sqrt{p}}{\sqrt{p+2i}}\cdot H(t)$$

or $$\tag{23}$$

$$f(t)=\frac{1}{\sqrt{2}}\cdot e^{i\left(t-\frac{\pi}{4}\right)}\cdot\sqrt{p}\cdot\left(1-i\cdot\frac{p}{2}\right)^{-\frac{1}{2}}\cdot H(t).$$

Then we expand about the origin according to the binomial formula:

$$(1-i\cdot x)^{-\frac{1}{2}}=$$

$$1+\sum_{k=1}^{\infty}(-1)^{k}\cdot\frac{1\cdot3\cdot5\cdots(4k-1)}{2\cdot4\cdots(4k)}\cdot x^{2k}-i\cdot\sum_{k=1}^{\infty}(-1)^{k}\cdot\frac{1\cdot3\cdot5\cdots(4k-3)}{2\cdot4\cdots(4k-2)}\cdot x^{2k-1}. \tag{24}$$

Hence the asymptotic form of the term I_{q_1} looks:

$$I_{q_1}\backsim\frac{1}{\sqrt{2}}\cdot e^{i\left(t\ \frac{\pi}{4}\right)}.$$

$$\cdot\left\{\sqrt{p}+\sum_{k=1}^{n}(-1)^{k}\cdot\frac{1\cdot3\cdot5\cdots(4k-1)}{2\cdot4\cdots(4k)}\cdot\frac{1}{2^{2k}}\cdot p^{2k+\frac{1}{2}}\right. \tag{25}$$

$$\left.-i\cdot\sum_{k=1}^{n}(-1)^{k}\cdot\frac{1\cdot3\cdots(4k-3)}{2\cdot4\cdots(4k-2)}\cdot\frac{1}{2^{2k-1}}\cdot p^{2k-\frac{1}{2}}\right\}\cdot H(t).$$

Further we have:

$$p^{\frac{1}{2}}\cdot H(t)=\frac{t^{-\frac{1}{2}}}{\Pi\left(-\frac{1}{2}\right)}\cdot H(t)=\frac{1}{\sqrt{\pi t}}\cdot H(t),$$

$$p^{\frac{3}{2}}\cdot H(t)=p\cdot p^{\frac{1}{2}}\cdot H(t)=\frac{d}{dt}\cdot\frac{1}{\sqrt{\pi t}}\cdot H(t)=-\frac{1}{2}\cdot\frac{1}{\sqrt{\pi t}}\cdot\frac{1}{t}\cdot H(t),$$

and generally, if $n\geq1$,

$$p^{n+\frac{1}{2}}\cdot H(t)=(-1)^{n}\cdot\frac{1\cdot3\cdot5\cdots(2n-1)}{2^{n}}\cdot\frac{1}{\sqrt{\pi t}}\cdot\frac{1}{t^{n}}\cdot H(t).\quad\ldots \tag{26}$$

Accordingly we get the following explicit expression for the function $I_{q_1}(t)$ (The factor $H(t)$, of course, may be dropped):

$$I_{q_1} \backsim \frac{1}{2}\cdot\sqrt{\frac{2}{\pi t}}\cdot e^{i\left(t-\frac{\pi}{4}\right)}\cdot\left\{1+\sum_{k=1}(-1)^k\cdot\frac{[1\cdot3\cdot5\cdots(4k-1)]^2}{2\cdot4\cdots(4k)}\cdot\frac{1}{2^{4k}}\cdot\frac{1}{t^{2k}}+\right. \tag{27}$$

$$\left.+i\cdot\sum_{k=1}^{n}(-1)^k\cdot\frac{[1\cdot3\cdot5\cdots(4k-3)]^2}{2\cdot4\cdots(4k-2)}\cdot\frac{1}{2^{4k-2}}\cdot\frac{1}{t^{2k-1}}\right\}.$$

For brevity we will write

$$I_{q_1}(t) \backsim \frac{1}{2}\sqrt{\frac{2}{\pi t}}\cdot e^{i\left(t-\frac{\pi}{4}\right)}\cdot[A(t)+iB(t)]. \quad \ldots\ldots\ldots\ldots \tag{28}$$

Correspondently we obtain the contribution from the other branch-point $q_2=-i$; obviously this must be the conjugate form of I_{q_1}, i. e.

$$I_{q_2}(t) \backsim \frac{1}{2}\sqrt{\frac{2}{\pi t}}\cdot e^{-i\left(t-\frac{\pi}{4}\right)}\cdot[A(t)-iB(t)] \quad \ldots\ldots\ldots\ldots \tag{28 a}$$

At last, combining the expressions I_{q_1} and I_{q_2}, we arrive at the required asymptotic form of the Bessel function $J_0(t)$, viz:

$$J_0(t) \backsim \sqrt{\frac{2}{\pi t}}\cdot\left[A(t)\cdot\cos\left(t-\frac{\pi}{4}\right)-B(t)\cdot\sin\left(t-\frac{\pi}{4}\right)\right], \quad \ldots\ldots \tag{29}$$

where, according to (27), the coefficients $A(t)$ and $B(t)$ may be written:

$$\left.\begin{aligned} A(t)&=1+\sum_{k=1}^{n}(-1)^k\cdot\frac{[1\cdot3\cdot5\cdots(4k-1)]^2}{\underline{|2k}}\cdot\frac{1}{(8t)^{2k}},\\ B(t)&=\sum_{k=1}^{n}(-1)^k\cdot\frac{[1\cdot3\cdot5\cdots(4k-3)]^2}{\underline{|2k-1}}\cdot\frac{1}{(8t)^{2k-1}}.\end{aligned}\right| \quad \ldots\ldots \tag{30}$$

If only the first term of A is taken into account, the development of course degenerates into the formula (252), derived by a more heuristical reasoning.

We will not continue further the analysis of formula (29) but refer the reader to textbooks of Bessel functions and e. g. *Funktionentafeln von Jahnke-Emde* as to its managing in numerical calculus.

6. On the commutativity of the shifting integrator e—ap.

It follows from the definition integral of an integrational product that

$$e^{-ap}\cdot f(t)=e^{-ap}\cdot F(p)\cdot H(t)=F(p)\cdot e^{-ap}\cdot H(t)=f(t-a).\ \ldots\ (1)$$

As usual $f(t)$ is assumed to be of class nr 1 at least. Especially is

$$e^{-ap}\cdot H(t)=H(t-a).$$

Accordingly, the primary function of e^{-ap} has a point of discontinuity at $t=a$; if $a<0$ it does not vanish identically when $t<0$. Obviously this function is not of class 1, and there is no corresponding integral operator.

Now we shall prove the following important *proposition*, applicable in cases where e^{-ap} occurs combined with integral operators:

If $a>0$, the commutative law

$$e^{-ap}\cdot F\times\varphi(t)=F\times e^{-ap}\cdot\varphi(t)$$

holds for any function $\varphi(t)$ of class 1.

The proof runs:

$$R.\ M.=\left[f(0)+\int_0^t f'(t-\tau)\cdot d\tau\right]\times\varphi(t-a)$$

$$=f(0)\cdot\varphi(t-a)+\int_0^t f'(t-\tau)\cdot\varphi(\tau-a)\cdot d\tau.$$

$$L.\ M.=e^{-ap}\cdot\{F\times\varphi(t)\}=\left[f(0)\cdot\varphi(t)+\int_0^t f'(t-\tau)\cdot\varphi(\tau)\cdot d\tau\right]_{t-a}$$

$$=f(0)\cdot\varphi(t-a)+\int_0^{t-a} f'(t-a-\tau')\cdot\varphi(\tau')\cdot d\tau'$$

or, if we put $\tau'=\tau-a$,

$$L.\ M.=f(0)\cdot\varphi(t-a)+\int_a^t f'(t-\tau)\cdot\varphi(\tau-a)\cdot d\tau.$$

But now, a being positive, we have $\varphi(\tau - a) \equiv 0$ in the interval $0 < \tau < a$, and accordingly we may write as well

$$L.\, M. = f(0) \cdot \varphi(t - a) + \int_0^t f'(t - \tau) \cdot \varphi(\tau - a) \cdot d\tau = R.\, M.$$

Remark. In Chapter 6 we showed a certain parallelism between the symbolic method and the integrational calculus. Thus the exponential factor $e^{i\omega t}$ was seen to correspond to the unit-function $H(t)$. The shifting integrator e^{-ap} has its analogon in the exponential factor $e^{-a \cdot i\omega}$; in fact, F denoting the complex vector representing e. g. the harmonic function

$$f(t) = Im \{F \cdot e^{i\omega t}\},$$

we get $f(t - a)$ represented by the vector $e^{-a \cdot i\omega} \cdot F$. For that reason the multiplying by e^{-ap} may be characterized as an operation of *phase-shifting.*

7. Extended theory of integral operators.

In Chapters 3 and 4 have been defined the four fundamental algebraical operations with integral operators. Accordingly, we are in a position of forming arbitrary *rational* functions of integral operators, e. g. such a function as

$$\frac{a \cdot F^2 + b \cdot H \times K}{c \cdot \Phi^2 \times \Psi + d},$$

where the capital letters may denote operators, the others ordinary numbers. However, the procedure of division, being essentially equivalent to the solution of a Volterra equation, will as a rule lead to not closed expressions.

There may be reason for developing to some more extent the parallelism between the theory of basic integrators and that of integral operators. We will start with deducing the operators of the successive derivatives of a function. The operators of $f(t)$, $f'(t) \cdots f^{(n)}(t)$ may be denoted by F, $'F, \cdots {}^{(n)}F$.

Let $\varphi(t)$ be any function and form the product

$$\psi(t)=F\times\varphi(t)=f(0)\cdot\varphi(t)+\int_0^t f'(t-\tau)\cdot\varphi(\tau)\cdot d\tau.$$

By differentiation:

$$\frac{d}{dt}F\times\varphi(t)=f(0)\cdot\frac{d}{dt}\varphi(t)+\left[f'(0)+\int_0^t f''(t-\tau)\cdot d\tau\right]\times\varphi(t).$$

But, by our manner of writing:

$$\left[f'(0)+\int_0^t f''(t-\tau)\cdot d\tau\right]={}'F,$$

and accordingly we must have, $\varphi(t)$ being arbitrary,

$$'F=\frac{d}{dt}F-\frac{d}{dt}\cdot f(0).\ \dots\dots\dots\dots\dots\dots\dots\ (1)$$

The next step of differentiation gives:

$$''F=\frac{d}{dt}\left(\frac{d}{dt}F-\frac{d}{dt}\cdot f(0)\right)-\frac{d}{dt}\cdot f'(0),\text{ from which follows}$$

$$''F=\frac{d^2}{dt^2}F-\frac{d^2}{dt^2}\cdot f(0)-\frac{d}{dt}\cdot f'(0),$$

and it is easily proved by induction that

$$^{(n)}F=\frac{d^n}{dt^n}F-\left[\frac{d^n}{dt^n}\cdot f(0)+\frac{d^{n-1}}{dt^{n-1}}\cdot f'(0)+\dots\dots+\frac{d}{dt}\cdot f^{(n-1)}(0)\right],\dots(2)$$

which equality is formally equivalent to formula (20) of Chapter 2.

From the identities:

$$'F\times\varphi(t)=\frac{d}{dt}[F-f(0)]\times\varphi(t),$$

$$''F\times\varphi(t)=\frac{d}{dt}['F-f'(0)]\times\varphi(t),$$

. . . .

we may conclude the following statement:

From a function $^{(n-1)}F\times\varphi(t)$ to the function $^{(n)}F\times\varphi(t)$ one will always come on differentiating a function with the initial value zero.

This fact will be of importance in the applications because of the following *commutative law*:

The product

$$\frac{d}{dt}\cdot F = F \times \frac{d}{dt},$$

i. e. is commutative, provided that it operates upon a function vanishing at the origin.

For the proof we have to compare the two functions

$$\psi_1(t) = \frac{d}{dt} F \times \varphi(t) \quad \text{and} \quad \psi_2(t) = F \times \frac{d}{dt}\varphi(t).$$

According to the foregoing:

$$\psi_1(t) = f(0)\cdot\varphi'(t) + {}'F \times \varphi(t).$$

Further we get by integration by parts:

$$\psi_2(t) = f(0)\cdot\varphi'(t) + \int_0^t f'(t-\tau)\cdot\varphi'(\tau)\cdot d\tau$$

$$= f(0)\cdot\varphi'(t) + [f'(t-\tau)\cdot\varphi(\tau)]_0^t + \int_0^t f''(t-\tau)\cdot\varphi(\tau)\cdot d\tau$$

$$= f(0)\cdot\varphi'(t) - f'(t)\cdot\varphi(0) + {}'F \times \varphi(t).$$

Hence $\psi_1(t) = \psi_2(t)$, only provided that $\varphi(0) = 0$.

Now we will apply the operator $I = [\int_0^t d\tau]$ to both members of (3). Because of the lemma just stated we get

$$I \times {}'F \times \varphi(t) = I \times \frac{d}{dt}[F - f(0)] \times \varphi(t) = \frac{d}{dt}\cdot I \times [F - f(0)] \times \varphi(t)$$

$$= [F - f(0)] \times \varphi(t),$$

and, $\varphi(t)$ being arbitrary, this necessitates that

$$I \times {}'F = F - f(0). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

By repeated use of this important rule we get

$$I^n \times {}^{(n)}F = F - [f(0) + I\cdot f'(0) + \cdots + I^{n-1}\cdot f^{(n-1)}(0)], \;\ldots\ldots\; (6)$$

which equality ought to be compared with (2).

Formula (6) furnishes us with a means of getting by purely operational calculus the solution of a linear differential equation with constant coefficients under given initial conditions. The procedure may be exemplified on the simple case:

$$\left.\begin{aligned} &\frac{d^2x}{dt^2} + 2\beta\cdot\frac{dx}{dt} + \omega^2\cdot x = f(t),\\ &x(0) = u,\;\; \frac{dx}{dt}(0) = v. \end{aligned}\right|$$

If X and F denote the integral operators of $x(t)$ and $f(t)$ respectively, the first equation may be written

$$''X \times H(t) + 2\beta\cdot{}'X \times H(t) + \omega^2\cdot X \times H(t) = F \times H(t).$$

Obviously the law of uniqueness must subsist also for integral operators, and hence we get for X the "differential" equation

$$''X + 2\beta\cdot{}'X + \omega^2\cdot X = F. \;\ldots\ldots\ldots\ldots\ldots\ldots\; (7)$$

This may be solved by multiplying by I^2 and then employing formula (6). Thus the equation will appear in the "algebraized" form:

$$[1 + 2\beta\cdot I + \omega^2\cdot I^2] \times X = I^2 \times F + u + (v + 2\beta u)\cdot I. \;\ldots\ldots\; (8)$$

The operational factor before X may be written:

$$L \equiv 1 + 2\beta\cdot I + \omega^2\cdot I^2 = \left[1 + \int_0^t \{2\beta + \omega^2\cdot(t-\tau)\}\cdot d\tau\right]. \;\ldots.\; (9)$$

The corresponding inverse operator L^{-1} can always be obtained according to the general method (i. e. by solution of a Volterra equation), and we get on multiplying (8) by L^{-1}:

$$X = L^{-1} \times I^2 \times F + u\cdot L^{-1} + (v + 2\beta u)\cdot L^{-1} \times I.$$

Hence the explicit time function becomes

$$\left.\begin{aligned} x(t)&=x_1(t)+x_2(t),\\ x_1(t)&=L^{-1}\times I^2\times f(t),\\ x_2(t)&=[u+(v+2\beta u)\cdot I]\times L^{-1}\times H(t). \end{aligned}\right\} \quad\cdots\cdots\cdots (10)$$

This result should be compared with the integrational solution (83) in Chapter 6 of the same problem; the expressions agree formally, if p be interpreted as I^{-1}.

At last we are going to examine the inverse operation of $F\times\varphi(t)$ in the case of degeneration where $f(0)=0$, thus

$$F=\left[\int_0^t f'(t-\tau)\cdot d\tau\right]. \quad\cdots\cdots\cdots\cdots\cdots (11)$$

Let $\varphi(t)$ be any function of class 1 and form

$$\psi(t)=F\times\varphi(t)=\int_0^t f'(t-\tau)\cdot\varphi(\tau)\cdot d\tau. \quad\cdots\cdots\cdots (12)$$

Proposed that the primary function $f(t)$ belongs to class 2 at least, we can differentiate both members of (12), thus getting

$$f'(0)\cdot\varphi(t)+\int_0^t f''(t-\tau)\cdot\varphi(\tau)\cdot d\tau=\psi'(t)$$

or

$$'F\times\varphi(t)=\frac{d}{dt}\psi(t). \quad\cdots\cdots\cdots\cdots\cdots\cdots (13)$$

Now, if $f'(0)\neq 0$, we know the inverse operator of $'F$ to exist and may thus apply $('F)^{-1}$ to both sides of (13). Because $\psi(0)=0$, we obtain

$$\varphi(t)=('F)^{-1}\times\frac{d}{dt}\psi(t)=\frac{d}{dt}\cdot('F)^{-1}\times\psi(t). \quad\cdots\cdots\cdots (14)$$

Accordingly, when $f(0)=0$, we may define the inverse operator of F by the equality

$$F^{-1}=('F)^{-1}\times\frac{d}{dt}=\frac{d}{dt}\cdot('F)^{-1}. \quad\cdots\cdots\cdots\cdots\cdots (15)$$

This must always have the character of an *integro-differential* operator.

If also $f'(0)=0$, while $f''(0) \neq 0$ and $f(t)$ is of class 3 at least, we may take another step upon the same way, defining in (15) $('F)^{-1}$ by the expression

$$('F)^{-1}=(''F)^{-1}\times\frac{d}{dt}=\frac{d}{dt}\cdot(''F)^{-1}.$$

Hence it follows that

$$F^{-1}=(''F)^{-1}\times\frac{d^2}{dt^2}=\frac{d^2}{dt^2}\cdot(''F)^{-1},$$

and we may formulate the *general rule:*

If the function $f(t)$ belongs to class $(n+1)$ at least and if $f(0)=f'(0)=\cdots=f^{(n-1)}(0)=0$, $f^{(n)}(0) \neq 0$, the inverse operator of the integral operator F is defined by the formula

$$F^{-1}=({}^{(n)}F)^{-1}\times\frac{d^n}{dt^n}=\frac{d^n}{dt^n}\cdot({}^{(n)}F)^{-1}. \qquad (16)$$

Accordingly, $\psi(t)$ being a sufficiently regular function, we are now in a position of solving the operational equation

$$F\times\varphi(t)=\psi(t)$$

even in cases where $f(t)$ and some of its derivatives vanish at the origin.

It may be of interest to identify the inverse operator of the fundamental integral operator I with the derivative operator $\frac{d}{dt}$. For that purpose we have only to put $f(0)=0$, $f'(t)=H(t)$, thus getting $'I=('I)^{-1}=1$.

8. Note about the differentiation and the integration of a basic integrator with respect to a parameter.

Let $f(x,t)$ be a function of time and a parameter x (e. g. a space-coordinate) such that

$$f(x,t),\ \frac{\partial f}{\partial x}\ \text{and}\ \int_a^b f(x,t)\,dx$$

have basic integrators. Then the following propositions are immediate consequences of the integral theorem:

1. If

$$\frac{\partial}{\partial x}\int_0^\infty f(x,\tau)\cdot e^{-p\tau}\,d\tau=\int_0^\infty \frac{\partial f(x,\tau)}{\partial x}\cdot e^{-p\tau}d\tau, \quad\ldots\ldots\ldots\quad (1)$$

then

$$\frac{\partial}{\partial x}[F(x;p)\cdot H(t)]=\frac{\partial F(x;p)}{\partial x}\cdot H(t).$$

2. If

$$\int_a^b\left[\int_0^\infty f(x,\tau)\cdot e^{-p\tau}d\tau\right]dx=\int_0^\infty\left[\int_a^b f(x,\tau)\,dx\right]\cdot e^{-p\tau}d\tau, \quad\ldots\quad (2)$$

then

$$\int_a^b[F(x;p)\cdot H(t)]\,dx=\left[\int_a^b F(x;p)\,dx\right]\cdot H(t).$$

It may be leaved to the reader to establish these relations.

It is sufficient for the fulfilment of the conditions (1) and (2) that for the range of x-values under consideration the respective time functions belong to class 1. Indeed, then the necessary uniform convergence of the infinite integrals is guaranteed, p being situated in a suitable half-plane. Accordingly, we may state the *theorem:*

Any function $f(x, t)$ of class 1 may be differentiated or integrated with respect to the parameter x by differentiating resp. integrating the corresponding basic integrator $F(x; p)$ before its application to $H(t)$, provided that the differentiated resp. integrated time function belongs itself to class 1.

9. Table of principal formulas.*

1. $$\frac{1}{p^s}\cdot H(t)=\frac{t^s}{\Pi(s)}\cdot H(t),\ s>-1\ \ldots\ldots\ldots\ldots\quad (170)$$

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[*] This table presents only such formulas as have been deduced in the text. It may be referred to the indices 4, 5 and especially 8 of the Bibliography for a more extensive collection of operational formulas.

2.
$$\frac{p}{p-\alpha}\cdot H(t)=e^{\alpha t}\cdot H(t).$$

3.
$$\frac{p}{(p-\alpha)^n}\cdot H(t)=\frac{t^{n-1}}{\underline{|n-1}}\cdot e^{\alpha t}\cdot H(t).$$

4.
$$\frac{1}{p-\alpha}\cdot H(t)=\frac{1}{\alpha}(e^{\alpha t}-1)\cdot H(t).$$

5.
$$\frac{p^2}{p^2+\omega^2}\cdot H(t)=\cos\omega t\cdot H(t).$$

6.
$$\frac{\omega p}{p^2+\omega^2}\cdot H(t)=\sin\omega t\cdot H(t).$$

7.
$$p\cdot\frac{(p+\beta)\cdot\sin\varphi+\omega\cdot\cos\varphi}{(p+\beta)^2+\omega^2}\cdot H(t)=e^{-\beta t}\cdot\sin(\omega t+\varphi)\cdot H(t).$$

8.
$$\frac{\Phi(p)}{\Psi(p)}\cdot H(t)=\left[\frac{\Phi(0)}{\Psi(0)}+\sum_{\nu=1}^{N}\frac{\Phi(p_\nu)}{p_\nu\cdot\Psi'(p_\nu)}\cdot e^{p_\nu t}\right]\cdot H(t). \qquad (142)$$

9.
$$e^{-\alpha\cdot\sqrt{p}}\cdot H(t)=\left[1-\operatorname{erf}\left(\frac{\alpha}{2\sqrt{t}}\right)\right]\cdot H(t),\ \alpha\geq 0\ldots\ldots \quad (218)$$

10.
$$p\cdot e^{-\alpha\cdot\sqrt{p}}\cdot H(t)=\frac{1}{2\sqrt{\pi}}\cdot\frac{\alpha}{t\sqrt{t}}\cdot e^{-\frac{\alpha^2}{4t}}\cdot H(t)\ \ldots\ldots \quad (220)$$

11.
$$\sqrt{p}\cdot e^{-\alpha\cdot\sqrt{p}}\cdot H(t)=\frac{1}{\sqrt{\pi t}}\cdot e^{-\frac{\alpha^2}{4t}}\cdot H(t)\ldots\ldots\ldots \quad (221)$$

12.
$$\frac{p}{\sqrt{p^2+\omega^2}}\left(\frac{\sqrt{p^2+\omega^2}-p}{\omega}\right)^{\lambda}\cdot H(t)=J_\lambda(\omega t)\cdot H(t),\ \lambda\geq 0\ldots \quad (233)$$

13.
$$\frac{p}{\sqrt{p^2+\omega^2}}\cdot e^{-a\cdot\sqrt{p^2+\omega^2}}\cdot H(t)=$$
$$=\sum_{\nu=0}^{\infty}\frac{(-a\omega)^\nu}{\underline{|\nu}}\cdot J_\nu(\omega(t-a))\cdot H(t-a)=J_0(\omega\cdot\sqrt{t^2-a^2})\cdot H(t-a). \qquad (234,\ 241)$$

14. $$e^{ap} \cdot f(t) = f(t+a).$$

15. $$\Phi(p+a) \cdot f(t) = e^{-at} \cdot \Phi(p) \cdot e^{at} \cdot f(t).$$

15 a. $$\Phi(p) \cdot f(t) = e^{at} \cdot \Phi(p+a) \cdot e^{-at} \cdot f(t).$$

16. If $\boldsymbol{F}(p) \cdot \boldsymbol{H}(t) = f(t)$, then $\boldsymbol{F}(\lambda p) \cdot \boldsymbol{H}(t) = f\left(\frac{t}{\lambda}\right)$, $\lambda > 0$.

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